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Equivariant asymptotic dimension

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
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Streszczenie

Praca poświęcona jest tzw. ekwiwariantnemu wymiarowi asymptotycznemu („wide equivariant covers” z [3, 4]) wprowadzonemu w [3] w ramach dowodu hipotezy Farrella–Jonesa. Zestawiamy to nowe pojęcie z klasycznym wymiarem asymptotycznym Gromova oraz własnością „transfer reducibility” ([1]) i badamy jego własności. Wykazujemy w szczególności, że dla oszacowania ekwiwariantnego wymiaru asymptotycznego nie trzeba rozważać uzwarcenia przestrzeni, na której badana grupa działa geometrycznie, lecz wystarczy ograniczyć się do brzegu tego uzwarcenia. Dowodzimy także, że grupy wirtualnie cykliczne to dokładnie te grupy, dla których omawiany wymiar zeruje się. Formułujemy warunki równoważne wobec definicji, m.in. charakteryzację w terminach własności A Yu oraz ε-odwzorowań w kompleks symplicjalny. Wzmacniamy również wynik [10], konstruując „ekwiwariantne rozdrobnienia” dla nieskończonych grup.

Słowa kluczowe

wymiar asymptotyczny, grupa dyskretna, uzwarcenie, brzeg Gromova

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

51 Geometry
51F Metric geometry
20 Group theory and generalisations
20F Special aspects of infinite or finite groups

Tytuł pracy w języku polskim

Ekwiwariantny wymiar asymptotyczny

Contents

Introduction

Definition of equivariant asymptotic dimension occurs implicitly in Theorem 1.1, [3] and Assumption 1.4, [4]. The first of this articles proves finiteness of equivariant asymptotic dimension for hyperbolic groups in order for the other to derive the Farrell–Jones conjecture.

A similar, yet weaker property, "transfer reducibility" was justified for other classes of groups in subsequent works (including [1, 5]), as it is also sufficient for proving the Farrell–Jones conjecture. Moreover, it was used in [2] to show the Borel conjecture for hyperbolic and CAT(0) groups.

In this thesis, while remembering about motivation coming from applications in great conjectures of algebraic topology, we focus on geometric aspects of equivariant asymptotic dimension.

Chapters 1-3 are intended as a smooth introduction into the topic – we identify important elements of the notion, consider examples and compare equivariant asymptotic dimension with asymptotic dimension. Chapters 4-5 prove new results about equivariant asymptotic dimension and alternative characterisations. Chapter 6 is devoted to explaining definition of transfer reducibility and its relation with equivariant asymptotic dimension. Finally, Chapter 7 contains a result on "equivariant topological dimension".

More specifically, in Chapter 1 necessary conventions and notation are introduced and basic definitions provided. A brief comparison of the equivariant asymptotic dimension with the asymptotic dimension and transfer reducibility is done.

Roughly speaking, finiteness of equivariant asymptotic dimension of a group G is existence of arbitrarily big coverings behaving well with respect to the group action and with uniformly bounded dimension[1] (multiplicity). The G-space, which is covered, is Cartesian product of the group with a suitable compactification $\overline{X}$ of a space X admitting a geometric G-action.

In the next two chapters 2, 3, properties of coverings and compactifications are discussed. We prove that a covering a non-compactified space[2] X exists under very mild assumptions, which concern space X rather than group G. This result is used in subsequent considerations about geometry of the coverings.

In Chapter 3, we elaborate on – previously introduced – notion of asymptotic compactification in order to observe, why abelianity may be an obstacle for finiteness of equivariant asymptotic dimension. We provide simple examples of asymptotic and only partially asymptotic compactifications – some of them come from [1] and [3].

Chapter 4 contains two results about equivariant asymptotic dimension. Using mentioned above observations of Chapter 2, we prove that in order to construct a covering of $G \times \overline{X}$ it is enough to cover $G \times \partial X$. In case of the free group it means interchanging a compactification of a 4-regular tree with the Cantor set for the Cantor set itself.

The second result is a theorem saying that virtually cyclic groups are exactly the groups

[1] The minimal bound is called the equivariant asymptotic dimension of the group.

[2] Strictly speaking: covering of $G \times X$.

with equivariant asymptotic dimension equal to zero. Furthermore, the theorem remains true if we consider coverings of $G \times \partial X$.

Chapter 5 provides alternative characterisations of equivariant asymptotic dimension. Analogs of classic characterisations of asymptotic dimension are given. However, one natural analog seems stronger then the definition – we leave its hypothetical equivalence as an open problem. Moreover, an interesting description – previously known to the experts – of equivariant asymptotic dimension in terms of property A is formulated and its equivalence with the definition is shown.

In Chapter 6, we introduce the notion of transfer reducibility, provide some intuitions and prove (after [2]) that finiteness of equivariant asymptotic dimension implies transfer reducibility.

The last chapter strengthens the result of [10], which is an improvement necessary for the application of techniques of [3, 5] in the case of the proof of the Farrell–Jones conjecture for different linear groups in [16]. It shows – under appropriate assumptions – finiteness of the "equivariant topological dimension", given finiteness of topological dimension.

All ideas and results – unless explicitly stated otherwise – come from the author inspired by the supervisor, Dr Piotr Nowak, to whom the author is heartfully grateful.

Chapter 1

The basics

1.1. Conventions

Notation 1.1.1. x^r will denote an open ball with centre x and radius r supposing the metric is obvious from the context. The metric neighbourhood $A^r = \bigcup_{x \in A} x^r$ is defined accordingly.

The group action will be denoted by the dot, e.g. "$g.x$" or without any extra signs: "gx".

When a group G acts on a topological space X (on a set Y), we will shortly say that X (Y) is a G-space (a G-set).

$G \backslash X$ will denote a quotient of the set (the topological space) X under the left action of G. A similar notation "$X \setminus Y$" will be used to denote the set-theoretical difference X minus Y, but the meaning should be clear from the context.

Sometimes we will write "for all $\alpha < \infty$..." to denote "for all $\alpha \in (0, \infty)$..." in order to clarify that the following condition is trivial for "small" α and needs justification for "big" ones (contrary to the situation in analysis or topology, where one often writes "for all $\varepsilon > 0$").

Definition 1.1.2. We say that a group action of G on a topological space X is *proper* if for every $x \in X$ there is its neighbourhood U such that the set $\{g \mid g.U \cap U \neq \emptyset\}$ is finite.

The group action is *cocompact* if the quotient space $G \backslash X$ is compact.

Convention 1.1.3. Throughout this thesis (unless explicitly stated otherwise) we will assume that G is a finitely generated group with a fixed word-length metric d_G. The group G acts properly and cocompactly by isometries on a metric space (X, d_X). By $(\overline{X}, d_{\overline{X}})$ we will denote a metrisable compactification of X admitting a continuous (not necessarily isometric) G-action compatible with the action on X. We will write ∂X to denote $\overline{X} \setminus X$.

1.2. Equivariant asymptotic dimension

Let us start with a few introductory definitions coming from [2] and [3].

Definition 1.2.1. A group H is *virtually cyclic* if it contains a cyclic subgroup of finite index.

In particular any finite group is virtually cyclic. We have the following elementary observations.

Proposition 1.2.2. *Virtually cyclic subgroups are finitely generated.*

The class of virtually cyclic subgroups is closed under taking subgroups and homomorphic images.

Infinite subgroups of virtually cyclic subgroups are of finite index.

Proof. Let G be virtually cyclic and $Z = \langle a \rangle$ be its cyclic subgroup of finite index.

Pick one element a_i from each coset of Z. Then the set $\{a_i\}_i \cup \{a\}$ generates G.

Let H be any subgroup of G. Assume that $H \cap Z \neq \{1\}$ – it is a cyclic subgroup of finite index in Z. So $H \cap Z$ is of finite index in G and thus in H. On the other hand, if $H \cap Z = \{1\}$, then H must intersect each coset of Z at most once, so it is finite. That shows that subgroups of virtually cyclic groups are virtually cyclic.

From the above we can also conclude that infinite subgroups of G are always of finite index.

Let $\phi : G \to I$ be a homomorphism. Note that $G = \bigcup_{i=1}^{n} a_i Z$, where a_i are as before. Since homomorphic images of cyclic groups are cyclic, we can present I as a finite sum of cosets of a cyclic subgroup as follows: $\bigcup_{i=1}^{n} \phi(a_i)\phi(Z)$. $\square$

In particular, we know that the set of virtually cyclic subgroups of a fixed group H satisfies the following definition. We will denote it as $\mathcal{VC}\text{yc}$.

Definition 1.2.3. A *family $\mathcal{F}$ of subgroups of the group H* is a set of subgroups of H closed under conjugation and taking subgroups.

Now we can define open $\mathcal{F}$-covers, crucial for the definition of equivariant asymptotic dimension.

Definition 1.2.4. Let Y be an H-space and $\mathcal{F}$ be a family of subgroups of H. A subset $U \subseteq Y$ is called an $\mathcal{F}$-subset if:

a) elements hU of the orbit of U are either equal or disjoint,

b) the stabiliser of U, thus the subgroup $H_U = \{h \in H \mid hU = U\}$ is a member of $\mathcal{F}$.

An open $\mathcal{F}$-cover of Y is a collection $\mathcal{U}$ of open $\mathcal{F}$-subsets of Y such that:

1) $\mathcal{U}$ covers Y: $\bigcup \mathcal{U} = Y$;

2) $\mathcal{U}$ is an H-set: $H\mathcal{U} = \mathcal{U}$.

The name "equivariant asymptotic dimension" comes from the fact that the coverings in its definition are $\mathcal{F}$-covers, thus by condition 2) above invariant (equivariant) under the group action.

Definition 1.2.5. For a family of subsets $\mathcal{U}$ of the set Y, by $\dim \mathcal{U}$ (dimension[1] of $\mathcal{U}$) we will denote the number $\sup_{x \in X} \#\{U \in \mathcal{U} \mid x \in U\} - 1$ (or infinity), where $\#A$ is the cardinality of A.

The following is a reformulation of definition appearing implicitly in theorem 1.1 from [3] and assumption 1.4 from [4].

Definition 1.2.6. Equivariant asymptotic dimension of a group G, denoted by eq-asdim G, is the smallest integer n such that there is a space X and its compactification $\overline{X}$ such that for every $\alpha < \infty$ there exists an open $\mathcal{VC}\text{yc}$-cover $\mathcal{U}$ of $G \times \overline{X}$ (with the diagonal G action) satisfying:

1. $\dim(\mathcal{U}) \leq n$,

2. for each $g \in G$ and $c \in \overline{X}$ there exists $U \in \mathcal{U}$ such that $g^{\alpha} \times \{c\} \subseteq U$,

[1] The definition and notation is motivated by relation between this number for suitable families $\mathcal{U}_\alpha$ and different notions of dimension (topological, asymptotic, equivariant asymptotic). This motivation also explains appearance of the term "-1" in the formula.

3. $G\backslash\mathcal{U}$ is finite.

We will call the coverings $\mathcal{U} = \mathcal{U}(\alpha)$ from the above definition eq-asdim-coverings and α-eq-asdim-coverings, if the constant α is important.

Definition 1.2.7. If a $\mathcal{VC}$yc-cover satisfies condition 2. from the above definition for a particular $\alpha > 0$, we will call α a G-Lebesgue number of this covering.

In fact, the third condition in the definition of eq-asdim is redundant.

Proposition 1.2.8. *Given an open $\mathcal{VC}yc$-cover $\mathcal{V}$ of $G \times \overline{X}$ satisfying conditions 1. and 2. (for a fixed α) from the definition of the equivariant asymptotic dimension, there exists a $\mathcal{VC}yc$-subcovering $\mathcal{U} \subseteq \mathcal{V}$ such that the third condition $\#(G\backslash\mathcal{U}) < \infty$ is satisfied.*

Proof. For a fixed $V \in \mathcal{V}$ let V_g denote the vertical section $V_g = \{x \in \overline{X} \mid (g,x) \in V\}$. Let $U(V) = \bigcap_{h \in 1^\alpha} V_h$ – this is an open set, as the intersection is finite and sections V_h are open. Moreover, the second condition implies that the family $\{U(V) \mid V \in \mathcal{V}\}$ covers $\overline{X}$. By compactness, we can find a finite $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $\{U(V_0) \mid V_0 \in \mathcal{V}_0\}$ is a finite subcover of $\overline{X}$. Thus for each $x \in \overline{X}$ there exists $V_0 \in \mathcal{V}_0$ containing[2] $1^\alpha \times \{x\}$. Clearly $1^\alpha \times \{x\} \subseteq V_0$ if and only if $h^\alpha \times \{hx\} \subseteq hV_0$, so the family $\mathcal{U} = G\mathcal{V}_0$ satisfies the second condition. By construction $G\backslash\mathcal{U}$ is finite and all the other conditions are satisfied, because $\mathcal{U}$ is a subset of $\mathcal{V}$. □

1.3. Relations to different notions

1.3.1. Relation of eq-asdim and asdim

The natural question which comes to mind is how the equivariant asymptotic dimension is related to the asymptotic dimension. Recall the definition.

Definition 1.3.1. Asymptotic dimension of a metric space G is the smallest integer such that for all $\alpha < \infty$ there is an open[3] covering $\mathcal{U}$ of G such that:

1. $\dim(\mathcal{U}) \leq n$,
2. for each $g \in G$ there exists $U \in \mathcal{U}$ such that $g^\alpha \subseteq U$,
3. $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$ (uniform boundedness)[4].

Coverings $\mathcal{U} = \mathcal{U}(\alpha)$ from the above definition will be called asdim-coverings.

We can see that the first two points in the definition of asdim are very similar to the first two points in the definition of eq-asdim.

However, it seems that eq-asdim is a more subtle notion – to the best of our knowledge, the only class of groups which are known to be of finite equivariant asymptotic dimension, is the family of hyperbolic groups (see [3]) and the fact that they also have finite asymptotic dimension is classic and can be proven in a two-page article ([14]). The difficulty with eq-asdim arise (see section 3.1) already in the case of the simplest non-hyperbolic group – $\mathbb{Z}^2$, which can be immediately proven to be of asymptotic dimension 2. Moreover, in 2.2 and 2.3, we will discuss that the two notions are not very close to each other and it is not clear how to derive finiteness of asdim from the finiteness of eq-asdim.

[2] Actually it contains the product of a ball 1^α and an open neighbourhood of x, namely $U(V_0)$.

[3] Openness is optional in the definition.

[4] By $\operatorname{diam} U$ we denote the diameter of the set U.

To the extent of author's knowledge, finding a group of infinite equivariant asymptotic dimension is an open problem. Such a group could shed some light on the relation. It may be valuable to check if the equivariant asymptotic dimension of infinite-dimensional (in the asymptotic sense) groups must me infinite.

1.3.2. Relation of eq-asdim and transfer reducibility

It is worth pointing out that there is a number of papers proving the Farrell–Jones conjecture for particular classes of groups that use properties similar or related to finiteness of eq-asdim, yet not proving finiteness of eq-asdim itself for these classes.

For example in [1] the authors prove that so called CAT(0)-groups[5] are *transfer reducible* over a family of virtually cyclic subgroups. We will recall that definition in Section 6 and explain after [2, Proposition 2.1.] that essentially the notion of transfer reducibility over $\mathcal{VC}$yc is weaker than the finiteness of eq-asdim. The fact that the authors of [1, 5] show only transfer reducibility for CAT(0) and linear groups – while proving finiteness of eq-asdim for hyperbolic groups in [3] – may suggest that the notion is quite strong. The difficulty with equivariant dimension for CAT(0)-groups is in a way suggested in the introduction to [1, Chapter 3.]:

> It [group G] will act on a large ball in X. (The action of G on the bordification $\overline{X}$ is not suitable, because it has too large isotropy groups.)

– we make this suggestion more precise in Section 3.1. The quotation also explains the main difference between the finiteness of equivariant dimension and transfer reducibility – while the first requires one space $\overline{X}$ to be suitable for all α, the second allows us to pick different spaces for different parameters (and eases the requirement on the action of G to be only a *homotopy action*), see Definition 6.0.14 and Remark 6.0.15.

[5] CAT(0)-groups are defined as groups admitting a proper and cocompact isometric action on a CAT(0)-space.

Chapter 2

Geometry of coverings

2.1. Importance of compactness of $\overline{X}$

It turns out that the compactness of ∂X is crucial to the notion of eq-asdim.

Proposition 2.1.1. *If we replace the compactification $\overline{X}$ in the Definition 1.2.6 of* eq-asdim *by the space X itself (in other words: if we require coverings of $G \times X$ instead of $G \times \overline{X}$), then the obtained notion of dimension is trivial; i.e., always equal to zero.*

Proof. Let $X = G$ with the natural left action by multiplication. A good eq-asdim-covering for $G \times X$ is $\mathcal{U} = \{G \times \{x\} \mid x \in X\}$, which is of dimension 0, has infinite G-Lebesgue numbers and finite quotient $G \backslash \mathcal{U}$ (a singleton). Of course $\mathcal{U}$ is an open covering, as $G \times X$ is a discrete space. It is a $\mathcal{VC}$yc-cover, because:

- it is invariant under the action of G;
- $g.(G \times \{x\}) \cap G \times \{x\} = G \times \{gx\} \cap G \times \{x\} = \emptyset$, assuming that $g \neq 1$;
- the isotropy groups of each $U \in \mathcal{U}$ are trivial (essentially for the same reason as above). $\square$

The above proof (even though for a slightly modified definition) is an example to the remark stated in [3] about equivariant asymptotic dimension saying that the members of an eq-asdim-covering have to be large only in G-coordinate and in X-coordinate they may be really small. Making them X-small enabled us to assure that elements of the cover are indeed $\mathcal{VC}$yc-subsets (their translations are either equal or disjoint and the stabiliser is virtually cyclic).

The following proposition generalises the above one. Note that the class of spaces in point *a*) of the proposition covers the generality of definition of eq-asdim in the original formulation[1] in [3] and points *b*) and *b′*) cover the case of a Rips complex, which is actually considered therein.

Proposition 2.1.2. *Assume that a finitely generated group G acts on a topological space X cocompactly (we do not require convention 1.1.3). There is an* eq-asdim*-covering (with $\alpha = \infty$) of the space $G \times X$ under any of the following conditions:*

a) X is a simplicial complex, the action of G is simplicial and the simplex stabilisers are virtually cyclic[2],

[1] There, the space X is a simplicial complex with an isometric, simplicial, proper, cocompact G-action.

[2] This point can be generalised to G-CW-complexes, so CW-complexes admitting a G-action inherited from a G-action on the disjoint sum of cells inducing the CW-structure. A proof is the same.

b) X is metrisable and the G-action is proper;

b') X is locally compact and the G-action is proper.

Proof. Ad *a*). Since the action is cocompact, the complex is finite dimensional and there is a finite number of orbits of simplices. Let $\{\Delta_i\}_{i\in I}$ be representatives of the orbits, and let $I(d)$ denote the indices of d-dimensional simplices Δ_i. For each relatively open d-dimensional simplex Δ_i^o one can choose[3] an open neighbourhood $N(\Delta_i^o)$ such that all these neighbourhoods and their translations will be disjoint $\mathcal{VC}$yc-subsets (with stabiliser of $N(\Delta_i^o)$ equal to the stabiliser of Δ_i^o).

The set $\{G \times N(\Delta_i^o)\}_{i\in I}$ generates (under the action of G) an open equivariant cover $\mathcal{U}$ of $G \times X$. We claim that it is an eq-asdim-covering. Clearly $G\backslash\mathcal{U}$ is finite and $\mathcal{U}$ has infinite Lebesgue number in the G-coordinate. Because each $N(\Delta_i^o)$ is a $\mathcal{VC}$yc-set, then so are the sets in the covering. The dimension of the covering is equal to the dimension of X.

Ad *b*), *b'*). For each $x \in X$ we will construct its neighbourhood U_x being a $\mathcal{VC}$yc-subset. By properness of the action[4] we can find a neighbourhood U_x^0 disjoint with completion $C_x = Gx \setminus \{x\}$ of x in its orbit Gx and such that the set $RS_x = \{g \mid gU_x^0 \cap U_x^0 \neq \emptyset\}$ is finite.

Then, using either metrisability or local compactness of X, we choose a smaller neighbourhood U_x^1, such that its closure $\overline{U_x^1}$ is contained in U_x^0 – in particular it is disjoint with C_x. But we have the equivalence

$$\forall_{g:gx\neq x} gx \notin \overline{U_x^1} \iff \forall_{g:gx\neq x} x \notin g\overline{U_x^1},$$

so the set $U_x^2 = U_x^1 \setminus \bigcup_{g:gx\neq x} g\overline{U_x^1}$ contains x. It is open, as the sum can be taken over a finite set RS_x without affecting the difference. What we achieved is emptiness of the intersection $U_x^2 \cap gU_x^2 \subseteq \left(U_x^1 \setminus gU_x^1\right) \cap gU_x^1 = \emptyset$ for $gx \neq x$.

To handle the case $gx = x$, we set $U_x = \bigcap_{g:gx=x} U_x^2$. The intersection is finite (as the stabiliser of x is a subset of RS_x), so we just obtained a neighbourhood of x, which is a $\mathcal{VC}$yc-subset (with the stabiliser equal to the stabiliser of x).

Now recall that the quotient map $X \xrightarrow{q} G\backslash X$ is open and the quotient is compact. So $\{q(U_x)\}_{x\in X}$ is an open covering of a compact set and there is a finite family $x_1, \dots, x_n$ such that $\{q(U_{x_i})\}_{1\leq i\leq n}$ covers $G\backslash X$ and thus $\mathcal{U}_0 = \{gU_{x_i} \mid g \in G,\ 1 \leq i \leq n\}$ covers X. Clearly, the dimension of $\mathcal{U}_0$ is at most n and the family $\mathcal{U} = \{G \times U_0 \mid U_0 \in \mathcal{U}_0\}$ is an eq-asdim-covering of $G \times X$ for any $\alpha \leq \infty$. ☐

Note that in the last propositions we did not assume that G has finite asymptotic dimension. Our coverings consisted of sets of the form $G \times U$ for some $U \subseteq X$, so actually we produced G-invariant coverings of X. The properties we used were rather small-scale ones (like dimension of a simplicial complex).

2.2. How do asdim-coverings and eq-asdim-coverings differ?

Comparing Definition 1.2.6 of eq-asdim and 1.3.1 of asdim, we can see that there is no exact equivalent of uniform boundedness in the list of requirements for eq-asdim. Additional elements are: finiteness[5] of $G\backslash\mathcal{U}$ and the fact that $\mathcal{U}$ is a $\mathcal{VC}$yc-cover. Can we directly obtain an asdim-covering, which has to be uniformly bounded, from an eq-asdim-covering? Is it true, that $\operatorname{asdim} G \leq \text{eq-asdim}\, G$?

The answer is in general negative.

[3]We omit the details. A reader unfamiliar with such arguments is referred for example to [10, Lemma 3.4].

[4]And T_1-property, which is always assumed.

[5]Which is actually trivial, as we noticed in 1.2.8.

Example 2.2.1. Note that for $X = \mathbb{R}$ and $G = \mathbb{Z}$ acting on X by translations and any $\overline{X}$, for example $\overline{X} = [-\infty, +\infty]$, the one-element covering $\{G \times \overline{X}\}$ is an eq-asdim-covering for any α. Consequently

$$1 = \operatorname{asdim} \mathbb{Z} \nleq \text{eq-asdim}\, \mathbb{Z} = 0.$$

Thus we can see, that a method of obtaining an asdim-covering would have to deal with (refining) unbounded members of an eq-asdim-covering whose isotropy groups are infinite virtually-cyclic, and may have to increase the dimension of the covering.

We will continue this analysis in the next section.

2.3. Geometry of eq-asdim-coverings.

In this section, we will examine geometry of eq-asdim-coverings by looking at different embeddings of G into $G \times \overline{X}$ and pulling back this coverings to G. We will check if the pushed out covering resembles asdim-coverings.

We are going to consider[6]:

- $g \mapsto (g_0, gx_0) \mapsto gx_0$, which is a coarse equivalence if X is proper (see Lemma 2.3.1), and even a quasi-isometry if X is additionally quasigeodesic.
- $g \mapsto (g, gx_0)$, which is a coarse embedding[7], because metrics on both sides are proper and left-invariant,
- $g \mapsto (g, x_0)$ (trivial embedding),

Embedding of G into X: $g \mapsto gx_0$

Lemma 2.3.1. *The function $g \mapsto gx_0$ is a coarse equivalence for a proper X.*

Proof. We should check that the pseudometric induced on G from the metric on the right hand side is proper; i.e, any ball is finite. Properness of action implies[8,9] that $\#\{g \mid g.K \cap K \neq \emptyset\} < \infty$ for any compact K. Take a closed ball $\overline{B}(hx_0, r)$ centred a point of the orbit; indeed, the set

$$\{g \mid ghx_0 \in \overline{B}(hx_0, r)\} = h\{g \mid gx_0 \in \overline{B}(x_0, r)\}h^{-1} \subseteq h\{g \mid g.\overline{B}(x_0, r) \cap \overline{B}(x_0, r) \neq \emptyset\}h^{-1}$$

is finite by the above observation.

Now, the ball 1^r is finite, thus $\sup_{g \in 1^r} d_G(x_0, gx_0) < \infty$, so by left-invariance of the metric d_G and the induced pseudometric we can write

$$d_X(gx_0, g'x_0) \leq F(d_G(g, g'))$$

[6] We take $x_0 \in \overline{X}$ in the last two cases, $x_0 \in X$ in the first one.

[7] Actually, it is a quasi-isometric embedding, because

$$d_G(g, g') \leq d_{G \times X}\Big((g, gx_0), (g', g'x_0)\Big) \leq (1 + M) \cdot d_G(g, g'),$$

where $M = \max_{s \in S} d_X(x_0, sx_0)$ if S is a finite set of generators of G inducing the metric d_G.

[8] Consider a finite covering of K by balls x^{r_x} such that x^{2r_x} intersects finitely many of its translations. If K intersects infinitely many of its translations, then one of the balls x^{r_x} must intersect infinitely many translations of another ball y^{r_y}. Without loss of generality $r_x \leq r_y$. But then y^{2r_y} would intersect its translations infinitely many times, a contradiction.

[9] Is is even equivalent as X is locally compact.

for some F. On the other hand, because $\{g \mid gx_0 \in x^r\}$ is finite as we showed in the previous paragraph, we can write:

$$f(d_G(g, g')) \le d_X(gx_0, g'x_0)$$

for some f increasing to infinity.

So far we didn't use the fact that the action is cocompact and we obtained a coarse embedding. By cocompactness of the action (and local compactness of X) there is a compact set $K \ni x_0$ such that $G.K = X$ and thus Gx_0 is coarsely dense in X, so our coarse embedding is in fact a coarse equivalence. □

Corollary 2.3.2 (Milnor-Švarc lemma)**.** *The function $g \mapsto gx_0$ is a coarse equivalence for a proper and quasi-geodesic X.*

Proof. By the previous Lemma 2.3.1 it is enough to recall that groups with word-length metric are quasi-geodesic and any coarse equivalence of quasi-geodesic spaces is a quasi-isometry. □

The embedding $g \mapsto gx_0$ gives little hope of constructing an asdim-covering from an eq-asdim-covering, as we noticed in Section 2.1 that an eq-asdim-covering may be small in the X-coordinate. However, looking at it closer is very enlightening with respect to the geometry of eq-asdim-coverings.

Definition 2.3.3. A compactification $\overline{X}$ of a metric space X is asymptotic, if for each $x \in \partial X$ and its neighbourhood U and each $R < \infty$, there is a smaller neighbourhood V such that for all $y \in V \cap X$ the ball y^R is contained in U. We define asymptotic points $x \in \partial X$ of the compactification accordingly.

A lot of natural compactifications are asymptotic, we give some examples in 3.2 and 3.3. The condition should be quite intuitive as it guarantees for example that for a sequence $x_n \in X$ convergent to some point in ∂X and y_n asymptotic[10] to x_n, y_n converges to the same point.

Example 2.3.4. Assume that X is proper and consider an eq-asdim-covering, a vertical slice $\overline{X}_{g_0} = \{g_0\} \times \overline{X}$ and pull this covering back by the coarse embedding $G \to \overline{X}_{g_0}$ given by the formula $g \mapsto (g_0, gx_0)$ to obtain the covering $\mathcal{U}$. The covering $\mathcal{U}$ is not uniformly bounded if the compactification is asymptotic.

Indeed, denote by $\mathcal{U}$ the restriction to $\overline{X} \simeq \overline{X}_{g_0}$ of the eq-asdim-covering of $G \times \overline{X}$ and by $\mathcal{U}_X$ the restriction of $\mathcal{U}$ to $X \subseteq \overline{X} \simeq \overline{X}_{g_0}$. Any $U_X \in \mathcal{U}_X$ which comes from some $U \in \mathcal{U}$ intersecting ∂X, by the definition of an asymptotic compactification contains arbitrarily big balls, so its diameter is infinite. In particular, $\mathcal{U}_X$ is not uniformly bounded and the covering pulled back via the coarse equivalence is not uniformly bounded either.

Thus, an eq-asdim-covering restricted to a vertical slice has to contain elements with infinite diameter "near the boundary" (for an asymptotic compactification) and may be small "away from the boundary" – which proved useful in the proof of 2.1.1 and 2.1.2. The first of this features contradicts uniform boundedness of an asdim-covering and the second the property of having big Lebesgue number.

In the next section we will formalise the concepts of "near" and "away" from the boundary, divide the covering into the corresponding parts and notice that they are – up to a point – independent.

[10] Sequences x_n, y_n are asymptotic if $\sup_n d_X(x_n, y_n) < \infty$.

“Diagonal” embedding: $g \mapsto (g, gx_0)$, $x_0 \in X$

Let us divide any eq-asdim-covering $\mathcal{U}$ of $G \times \overline{X}$ into two parts:

$$\mathcal{U}^o = \{U \in \mathcal{U} \mid U \cap G \times \partial X = \emptyset\},$$

$$\mathcal{U}^\partial = \{U \in \mathcal{U} \mid U \cap G \times \partial X \neq \emptyset\}.$$

Proposition 2.3.5. *Assume that X satisfies any of the assumptions of 2.1.2*[11]*. Then for any α-*eq-asdim *covering $\mathcal{U}$ of $G \times \overline{X}$ the covering $\mathcal{U}_t \cup \mathcal{U}^\partial$ is also an α-*eq-asdim*-covering, where $\mathcal{U}_t$ is the covering from 2.1.2.*

Proof. As $\mathcal{U}^\partial$ covers $G \times \partial X$ and is an $\mathcal{VC}$yc-cover with G-Lebesgue number α and $\mathcal{U}_t$ covers $G \times X$ as a $\mathcal{VC}$yc-cover with infinite G-Lebesgue number, their union is a $\mathcal{VC}$yc-cover of $G \times \overline{X}$ with G-Lebesgue number α. The quotient $G \backslash \mathcal{U}_t$ is finite and so is $G \backslash \mathcal{U}^\partial$ (because it is a subset of $G \backslash \mathcal{U}$), so $G \backslash (\mathcal{U}_t \cup \mathcal{U}^\partial)$ is also finite. In the same way (but independently of α) we can bound $\dim(\mathcal{U}_t \cup \mathcal{U}^\partial)$ by $\dim \mathcal{U}_t + \dim \mathcal{U} + 1$. $\square$

This shows that $\mathcal{U}^o$ can always (under some mild assumptions) be assumed to be $\mathcal{U}_t$. Consequently, if $g \mapsto gx_0$ is a coarse equivalence (e.g. if X is proper) then – from the fact that the X-Lebesgue number of $\mathcal{U}_t$ can be assumed to be bounded – it follows that the Lebesgue number of the covering of G pulled back via the diagonal embedding will always be smaller then some constant. So $\mathcal{U}^o$ can not be directly used to derive asdim-coverings for G.

Note also, that if we do not impose any conditions on X then it can be equal to the group G itself and be covered trivially as in Proposition 2.1.1 – such a covering induces a covering by singletons on the diagonal.

Perhaps U^∂ will prove more useful? If $x_0 \in X$ is fixed, then the answer is negative. We can replace all the $U \in \mathcal{U}^\partial$ by sets of the form $U \setminus G.(\{1\} \times \{x_0\})$. That way $\mathcal{U}^\partial$ is disjoint with “the diagonal”, so useless for pulling back the covering. The change does not spoil any of the important properties of $\mathcal{U}$: invariance properties are preserved, $\mathcal{U}^\partial$ still covers $G \times \partial X$ and $\mathcal{U}^o$ (we assume $\mathcal{U}^o = \mathcal{U}_t$) covers $G \times X$. $\mathcal{U}^o$ assures big G-Lebesgue numbers on X and the change do not affect G-Lebesgue numbers for ∂X. Dimension of the covering may even be smaller after the change.

Trivial embedding: $g \mapsto (g, x_0)$

As in the previous section we can change $\mathcal{U}^o$ such that its elements are of the form $G \times U$, thus inducing a one-element covering on embedded G.

On one hand, this embedding seems the most appropriate for trials of obtaining an asdim-covering, since for the pull back of the covering the first two requirements for an asdim-covering (dimension and Lebesgue number) follow directly from the corresponding conditions for an eq-asdim-covering. However we would have to somehow bound the diameter of its elements (perhaps after dealing with infinite elements of the covering related to virtually cyclic subgroups, compare 2.2.1).

On the other hand, we will see in Section 4.1 that the problem of eq-asdim-coverings of $G \times \overline{X}$ can be reduced to the problem of coverings of $G \times \partial X$, so – provided that obtaining asdim-coverings from eq-asdim-coverings is possible – focusing on the space X itself may not be the most straightforward approach.

[11] This weakens our general assumptions from convention 1.1.3. Assumptions of the proposition cover the generality of [3].

Summary

In 2.3.5 we noticed that covering of $G \times X$ can be assumed to be quite trivial (and useless for obtaining asdim-coverings). The actual difficulty and interesting geometry of the notion of equivariant asymptotic dimension lies in the covering of the boundary $G \times \partial X$ (open in the whole $G \times \overline{X}$), denoted above as $\mathcal{U}^{\partial}$.

Propositions 2.1.2 and 2.3.5 will be important ingredients of Theorem 4.1.4 (main result of the mentioned above Section 4.1) saying that the whole geometry is hidden in covering of the boundary $G \times \partial X$ (open in $G \times \partial X$, not in $G \times \overline{X}$).

Chapter 3

Geometry of compactifications $\overline{X}$

3.1. The case of $G = \mathbb{Z}^n$

Lemma 3.1.1. *If there are* eq-asdim*-coverings of $G \times \overline{X}$ (for all $\alpha < \infty$), then in $\overline{X}$ there are no points whose stabiliser has a finitely generated not virtually cyclic subgroup*[1]*.*

Proof. Assume there is $x \in \overline{X}$ whose stabiliser contains a finitely generated $H = \langle S \rangle$ which is not virtually cyclic. For sufficiently big α the ball 1^α contains S, and the associated to α cover $\mathcal{U}$ contains a set U such that $1^\alpha \times \{x\} \subseteq U$.

We note that for $s \in S$: $s.(1^\alpha \times \{x\}) = s^\alpha \times \{x\}$ intersects nontrivially with $1^\alpha \times \{x\}$ and thus $s.U \cap U \neq \emptyset$, so s must stabilise U and thus the stabiliser of U contains $H = \langle S \rangle$, so is not virtually cyclic. □

Corollary 3.1.2. *If there are* eq-asdim*-coverings for $\mathbb{Z}^n \times \overline{X}$, $n \geq 2$, then the compactification $\overline{X}$ can not be asymptotic at any point.*

Proof. Assume the converse that there is an asymptotic point $x \in \partial X$ and $x_n \in X$ convergent to x. Then for any $g \in \mathbb{Z}^n$ the sequence gx_n is asymptotic to x_n and thus converges to x. Consequently $\mathbb{Z}^n$ stabilises x, contradicting Lemma 3.1.1. □

Example 3.1.3. *Let $G = \mathbb{Z}^2$ and $X = \mathbb{R}^2$ with the natural $\mathbb{Z}^2$ action.*

Suppose there is a vertical line L such that its closure in $\overline{X}$ is countable (and thus by [9] *is a closed interval or a circle) and that for a boundary point $l \in \bar{L} \setminus L$ there is a limit $\lim_{n\to\infty}(n, 0).l$ or $\lim_{n\to\infty}(-n, 0).l$. Then there is no* eq-asdim*-covering for $\mathbb{Z}^2 \times \overline{X}$ and $\alpha > 1$.*

Proof. We want to find a point $x \in \overline{X}$ whose stabiliser is the whole $\mathbb{Z}^2$ and obtain a contradiction by Corollary 3.1.2. Without loss of generality we will assume that there exists a limit $l' = \lim_{n\to\infty}(n, 0).l$. Note that $\{0\} \times \mathbb{Z}$ stabilises l and thus its images under $(n, 0)$ and finally l' as their limit. Obviously l' is also stabilised by $(1, 0)$ as:

$$(1, 0).l' = (1, 0). \lim(n, 0).l = \lim(n + 1, 0).l = l',$$

which ends the proof. □

Remark 3.1.4. *The above example can be naturally generalised to the case where $G = \mathbb{Z}^n$ and $X = \mathbb{R}^n$ for $n \geq 2$.*

[1] A slightly confusing formulation comes from the fact that there exist not virtually cyclic groups such that all their finitely generated subgroups are virtually cyclic. Such an example is $(\mathbb{Q}, +)$.

Surprisingly, it turns out that it is difficult to control the equivariant asymptotic dimension of groups $\mathbb{Z}^n$, because they are... abelian. On one hand, as we noticed in 3.1.1, stabilisers of points should be small. On the other, asymptotic compactifications are very intuitive. For abelian groups[2] these two conditions are mutually exclusive, because the left action coincides with the right action, which always produces asymptotic sequences converging – for asymptotic compactifications – to the same boundary point as the initial one. (We write about action on the group itself, but the group embeds equivariantly into X and it is a Lipschitz embedding [often (see 2.3.2) a quasi-isometry]).

3.2. Compactifications of $X = \mathbb{R}^n$ for $G = \mathbb{Z}^n$

$\mathbb{R}^n$ is a natural space equipped with a proper and cocompact isometric action of $\mathbb{Z}^n$. Let us look at some examples.

Example 3.2.1. The simplest compactification is the one-point compactification, but the point at infinity is stabilised by the whole group, so by 3.1.1 this compactification can not be used to show finiteness of eq-asdim $\mathbb{Z}^n$. It is an asymptotic compactification.

In [1] a compactification called "bordification" is analysed for CAT(0) spaces. We will not check the details (available in [7]) of construction but will recall the definition and look at some examples.

Definition 3.2.2. We assume that X is CAT(0). A geodesic ray in X is an isometric embedding $c : [0, \infty) \to X$, a generalised geodesic ray $c : [0, \infty) \to X$ might be an isometric embedding on some interval $[0, a]$ and constant on $[a, \infty)$. Fix $x_0 \in X$. The bordification $\overline{X}$ of X is a set of all generalised geodesic rays starting at x_0 (however it can be proved, that in fact $\overline{X}$ does not depend on the choice of x_0).

Let $\rho_r(c) = c(r)$ be a projection $\overline{X} \to \overline{B}(x_0, r) \subseteq X$. We define the topology (called the cone topology) on $\overline{X}$ by the basis consisting of sets of the form $\rho_r^{-1}(V)$ for all r and V being an open subset of $\overline{B}(x_0, r)$. Equivalently we can say that the topology is defined as the topology of the inverse limit of the inverse system consisting of the balls $\overline{B}(x_0, r)$ and projections similar to the above (for any $x \in X$ there is a unique generalised geodesic ray joining x_0 and x, so we can define the projection of x as the projection of this ray).

The boundary ∂X is called the visual boundary and may also be defined as the set of equivalence classes of asymptotic[3] geodesic rays (not necessarily starting at x_0).

Example 3.2.3. Bordification of $\mathbb{R}^n$ is homeomorphic with the closed ball $0^{\pi/2}$, where the homeomorphism restricted to $\mathbb{R}^n$ is given by $(r, \phi) \mapsto (\mathrm{arctg}(r), \phi)$ in the polar coordinates. We can see that this compactification is asymptotic at every point and the boundary points are stabilised by the whole group $\mathbb{Z}^n$. This exemplifies the phenomenon mentioned in the quotation from [1] cited in 1.3.2 saying that boundary points of the bordification have too big stabilisers.

Example 3.2.4. Another possible compactification of $\mathbb{R}^n$ is the cube being closure of the image of the homeomorphism $c : \mathbb{R}^n \to (-\pi/2, \pi/2)^n$ defined by the formula $f(x_1, \ldots, x_n) = (\mathrm{arctg}(x_1), \ldots, \mathrm{arctg}(x_n))$. It is asymptotic only at the vertices. Let $J = [-\pi/2, \pi/2]$. For a point in the interior of the face $J^k \times \{\pm\pi/2\} \times \ldots \times \{\pm\pi/2\}$ its stabiliser is $\{0\} \oplus \ldots \oplus \{0\} \oplus \mathbb{Z}^{n-k}$.

[2] Not virtually cyclic.

[3] Two geodesic rays c, c' are asymptotic if $\sup_{t \in \mathbb{R}_+} d_X(c(t), c'(t)) < \infty$. It is an equivalence relation.

Comment 3.2.5. Note the difference between the last example 3.2.4 and the penultimate example 3.2.3 – even if the final compactifications may seem similar at the first sight, they are deeply different. In the penultimate example the homeomorphism was "radial", which leaded to the boundary with the trivial group action. We could use a „radial" homeomorphism onto the cube with the same effect. In the last example the homeomorphism *is not* "radial" and we can see that the stabilisers are much smaller.

The difference might not be obvious at first, because the homeomorphism in 3.2.4 is radial for specific lines – it preserves the coordinate axis and "diagonals" (lines of the form $\{(r, r, 0, 0, r, 0, \dots) \mid r \in \mathbb{R}\}$). However – for example for $n = 2$ – any geodesic ray with image of the form $\{(r, ar) \mid r \in \mathbb{R}_+\}$ for $0 < a < 1$ converges to the vertex $(\pi/2, \pi/2)$, whereas for the radial situation it would converge to $(\pi/2, a \cdot \pi/2)$ on the right side of the square. On the other hand, elements of the family $\{(r, k) \mid r \in \mathbb{R}_+\}_{k\in\mathbb{R}}$ converge to the respective points $(\pi/2, \operatorname{arctg}(k))_{k\in\mathbb{R}}$ in 3.2.4, while they would have a common limit $(\pi/2, 0)$ in the radial case.

3.3. Compactifications of the Cayley graph of $\mathbb{F}_2$

In [7, Proposition 3.7] it is proven that the bordification (used in [1] for CAT(0) spaces) and compactification by attaching the Gromov boundary (used in [3] for Gromov hyperbolic spaces) are the same if we assume that the space in question is proper, geodesic, hyperbolic and CAT(0).

The Cayley graph of $\mathbb{F}_2$ satisfies all these conditions. We will describe its compactification following the Definition 3.2.2 of bordification.

Example 3.3.1. Elements of the boundary ∂X of $X = \mathrm{Cay}(\mathbb{F}_2)$ are just geodesic rays in $\mathrm{Cay}(\mathbb{F}_2)$ starting at 1. They can be identified with infinite words over the alphabet $\{a, a^{-1}, b, b^{-1}\}$ in the same way as elements of $\mathbb{F}_2$ have canonical representation as finite words. Interior points of edges can be denoted similarly using formal fractional exponents at the last position; e.g. $aba^{-1}bba^{-1/2}$.

So we can define C_w to be the set of all $x \in \overline{X}$, which have $w \in \mathbb{F}_2$ as a proper prefix. We can easily see that a basis of the topology on $\overline{X}$ is formed by open sets in $\mathrm{Cay}(\mathbb{F}_2)$ and cones C_w. By form of the basis we can easily see that this compactification is asymptotic.

Note that we can divide ∂X into 4 parts: infinite words starting with a, b, a^{-1}, b^{-1} respectively – each of this parts is a "quaternary Cantor set" ($3^{\mathbb{N}}$) and thus the whole boundary is homeomorphic to the Cantor set.

Let us find the metric on $\overline{X}$ generating the cone topology. We can introduce a partial order on $\overline{X}$ coming from the linear orders on images of geodesic rays starting at 1 induced from $[0, \infty)$[4]. For $x, y \in \overline{X}$ there exists their greatest common ancestor $A(x, y)$ – the biggest (with respect to the partial order) element being smaller or equal than x and y. Let d_0 be any finite metric on $[0, \infty]$ compatible with the topology – it induces a metric (denoted also by d_0 for simplicity) on images of geodesic rays starting at 1. We define the distance $d(x, y) = d_0(x, A(x, y)) + d_0(A(x, y), y)$ for incomparable x, y and $d(x, y) = d_0(x, y)$, when x, y are comparable. It can be easily checked that this function indeed is a metric.

It is straightforward to prove that the identity function between $\overline{X}$ with the cone topology (generated by the basis as above) and the metric topology induced by d is a homeomorphism.

[4]When restricted to $\mathbb{F}_2 \cup \partial X$ this partial order is a tree of height $\omega + 1$ in the sense of set theory.

Chapter 4

Coverings of the boundary and zero-dimensional coverings

4.1. It is enough to cover $G \times \partial X$

4.1.1. Topological approach

Recall the (simplified) definition of bordification from [7].

Definition 4.1.1. Let Y be a geodesic complete CAT(0) space and $y_0 \in Y$ be a fixed basepoint. Recall that a geodesic ray in Y is an isometric injection $c\colon [0,\infty) \to Y$. The set ∂Y called the *visual boundary* is the set of geodesic rays starting at y_0. The union $Y \cup \partial Y$ is denoted by $\widehat{Y}$ and called the bordification of Y.

Note that we can identify $y \in Y$ with a (unique) geodesic segment $[0, d(y, y_0)] \xrightarrow{s_y} Y$ joining y_0 and y and further with a generalised geodesic ray $c_y(r) = s_y\big(\min(r, d(y, y_0))\big)$. We have the projections $p_r^\infty\colon \widehat{Y} \to S(r)$ given by $p_r^\infty(c) = c(r)$ and similarly $p_r^R\colon S(R) \to S(r)$ given by $p_r^R(y) = c_y(r)$, where $S(r)$ is an r-sphere about y_0. Clearly $p_r^s \circ p_s^R = p_r^R$. We will omit the upper index and write p_r.

Definition 4.1.2. The cone topology on $\widehat{Y}$ is the topology with the basis consisting of some basis of the topology of Y and the family of "cones":

$$U_{y_0}(c, r, \varepsilon) = \{y \in \widehat{Y} \mid d(y, y_0) > r,\ d(p_r(y), c(r)) < \varepsilon\},$$

where $c \in \partial Y$, $r < \infty$ and $\varepsilon > 0$ and if $y \in \partial Y$ then we put $d(y, y_0) = \infty$.

The above construction relies on the basepoint y_0, but it is shown in [7], that the result is independent of it. Moreover, if Y is hyperbolic, proper and geodesic, then the compactification by the Gromov boundary and by the visual boundary agree ([7, Proposition 3.7]). In particular, this holds for $\mathrm{Cay}(\mathbb{F}_2)$.

Theorem 4.1.3. *Suppose that one of the assumptions of 2.1.2 hold, any geodesic segment in X can be extended to a geodesic ray and $\overline{X} = \widehat{X}$ is a bordification of X. Then existence of an α-*eq-asdim *covering $\mathcal{U}$ of $G \times \partial X$ implies existence of an α-*eq-asdim*-covering $\mathcal{V}$ of $G \times \overline{X}$ with* $\dim \mathcal{V} \leq \dim \mathcal{U} + n + 1$, *where n is the dimension of the covering from 2.1.2.*

Proof. It is enough to extend the covering $\mathcal{U}$ to a $\mathcal{VC}$yc covering $\mathcal{W}$ of a neighbourhood of ∂X without increasing its dimension and sum it (like in 2.3.5) with an α-eq-asdim-covering from Proposition 2.1.2.

Fix $x_0 \in X$. Take any $U \in \mathcal{U}$ and its section $U_g = U \cap \big(\{g\} \times \overline{X}\big) \subseteq \overline{X}$. The basis of the topology on ∂X consists of traces of elements of the basis for the cone topology. Let $V_g(U) = \bigcup_{c,r,\varepsilon} U_{gx_0}(c,r,\varepsilon)$, where the sum is taken over such (c,r,ε) that $U_{gx_0}(c,r,\varepsilon) \cap \partial X \subseteq U_g$. Note that $U_g = V_g(U) \cap \partial X$ (we formed an open set as a sum of basis sets). Let $V(U) = \bigcup_g \{g\} \times V_g(U)$.

Put $\mathcal{W} = \{V(U) \mid U \in \mathcal{U}\}$. Indeed, the family $\mathcal{W}$ extends $\mathcal{U}$, namely $U \subseteq V(U)$, so – if we restrict ourselves to $G \times \partial X$ – α is a G-Lebesgue number for $\mathcal{W}$. By construction, $V(gU) = gV(U)$, so it is G-equivariant and stabiliser of $V(U)$ contains the stabiliser of U. If we show that $U \cap U' = \emptyset \implies V(U) \cap V(U') = \emptyset$ (the converse implication is also trivially true), then we will know that $\dim \mathcal{W} = \dim \mathcal{U}$ and (with a little abuse of logic notation):

$$gV(U) \cap V(U) \neq \emptyset \implies gU \cap U \neq \emptyset \implies gU = U \implies V(gU) = V(U),$$

meaning that each $V(U)$ is a $\mathcal{VC}$yc-set, so $\mathcal{W}$ is a $\mathcal{VC}$yc-cover.

So let us take $U, U' \in \mathcal{U}$ with empty intersection and assume that $V(U), V(U')$ intersect nontrivially at (g,x). It means that $V_g(U)$, $V_g(U')$ intersect at x and finally, they contain $W = U_{gx_0}(r,c,\varepsilon)$, $W' = U_{gx_0}(r',c',\varepsilon')$ (respectively) intersecting at x (with $r_x = d(x_0,x) > \max(r,r')$). Thus we have $d(c_x(r), c(r)) < \varepsilon$ and $d(c_x(r'), c(r')) < \varepsilon'$.

Note that for any set $U_{gx_0}(R,C,\epsilon)$ its intersection with the boundary ∂X can be described as the inverse image of a ball $(p_R^\infty)^{-1}(C(R)^\epsilon)$. We obtain:

$$(p_{r_x}^\infty)^{-1}(x) \subseteq (p_{r_x}^\infty)^{-1}\Big((p_r^{r_x})^{-1}\big(p_r^{r_x}(x)\big)\Big) = (p_r^\infty)^{-1}(c_x(r)) \subseteq (p_r^\infty)^{-1}(c(r)^\varepsilon)$$

and similarly for (r',c',ε') meaning that

$$(p_{r_x}^\infty)^{-1}(x) \subseteq W \cap W' \subseteq U_g \cap U'_g = \emptyset,$$

a contradiction[1]. ☐

4.1.2. Metric approach

It turns out that the above ideas can be generalised, if we assume that $\overline{X}$ is metrisable. It is not a big cost as compactifications in [3] are metrisable and metrisability is anyway assumed in applications to the Farrell–Jones conjecture, compare [4, Theorem 1.1]. We do *not* need to assume anything more like CAT(0), completeness, existence of extensions of geodesic segments to geodesic rays, etc.

Theorem 4.1.4. *If any of the assumptions of 2.1.2 hold and $\overline{X}$ is metrisable, then existence of an α-*eq-asdim *covering $\mathcal{U}$ of $G \times \partial X$ implies existence of an α-*eq-asdim*-covering $\mathcal{V}$ of $G \times \overline{X}$ with* $\dim \mathcal{V} \leq \dim \mathcal{U} + n + 1$, *where n is the dimension of the covering from 2.1.2.*

Proof. Fix a metric d inducing the topology of $\overline{X}$. For any $U \in \mathcal{U}$ we will define $V_g(U)$ for all $g \in G$ similarly as in the previous proof and the family $\mathcal{W}$ consisting of sets $V(U) = \bigcup_g \{g\} \times V_g(U)$ will be an eq-asdim covering of a *neighbourhood of* ∂X. Together with a covering of $G \times X$ from 2.1.2 it makes an α-eq-asdim-covering of $G \times \overline{X}$.

Fix $U \in \mathcal{U}$ and denote by U_g the section $\{x \in \partial X \mid (g,x) \in U\}$ and by $\widetilde{U}_g$ its translation $\widetilde{U}_g = g^{-1}U_g = (g^{-1}U)_1$. We need a somewhat awkward construction because we do not assume that d is G-invariant. For an open subset O of ∂X and $x \in O$, let $r_x(O) > 0$ be the

[1] We need the technical assumption that geodesic segments can be extended to geodesic rays to conclude that $(p_{r_x}^\infty)^{-1}(x)$ is not empty.

distance from x to the complement of O, namely $\partial X \setminus O$. For any $g \in G$ we define $\widetilde{V}_g(U) \subseteq \overline{X}$ by the following formula and $V_g = g\widetilde{V}_g$:

$$\widetilde{V}_g(U) = \bigcup \left\{ x^{r/2} \,\middle|\, x \in \widetilde{U}_g,\ r = r_x(\widetilde{U}_g) \right\}.$$

By construction, the covering $\mathcal{W} = \{V(U) \mid U \in \mathcal{U}\}$ is a G-equivariant open covering of some neighbourhood of $G \times \partial X$ with $gV(U) = V(gU)$. Again $V(U) \cap (G \times \partial X) = U$ (in particular $U \subseteq V(U)$), so the G-Lebesgue number at the boundary is preserved.

Like in the previous proof, it is enough to prove that $U \cap U' = \emptyset \implies V(U) \cap V(U') = \emptyset$, which, in turn, reduces to justifying that $V_g(U) \cap V_g(U') = \emptyset$ for all g. Assume the contrary – then there exists $y \in V_g(U) \cap V_g(U')$, in particular there are $x \in \widetilde{U}_g, z \in \widetilde{U}'_g$ such that $g^{-1}y \in x^{r_x/2} \cap z^{r_z/2}$, where $r_x = r_x(\widetilde{U}_g)$ and $r_z = r_z(\widetilde{U}'_g)$. Without loss of generality $r_z \le r_x$ and by the triangle inequality we have $d(x, z) < r_x/2 + r_z/2 \le r_x$, but then z would be in $\widetilde{U}_g$, a contradiction. □

4.2. Zero-dimensional is the same as virtually cyclic

Theorem 4.2.1. *Equivariant asymptotic dimension* eq-asdim $G = 0$ *if and only if the group* G *is virtually cyclic.*

Proof. If an infinite group is virtually cyclic, then it is quasi-isometric to $\mathbb{Z}$, thus a virtually cyclic group is hyperbolic and there exists (by [3]) X and $\overline{X}$ as in the definition of eq-asdim. It is enough to consider a one-element covering of $G \times \overline{X}$, like in example 2.2.1.

For the converse, assume that there is a $\mathcal{VC}$yc-cover $\mathcal{U}$ of $G \times \overline{X}$ of dimension 0; i.e., disjoint and G-Lebesgue number $\lambda > 1$. Take $U \in \mathcal{U}$ and $(g, x) \in U$. Then there exists $U' \in \mathcal{U}$ such that $g^\lambda \times \{x\} \subseteq U'$ – but then $U \cap U' \neq \emptyset$, so $U = U'$. Thus we showed that $(g, x) \in U$ implies $g^\lambda \times \{x\} \subseteq U$, so $G \times \{x\} \subseteq U$ and we conclude that $U = G \times U_{\overline{X}}$ for the open set $U_{\overline{X}} \subseteq \overline{X}$, $U_{\overline{X}} = \pi_{\overline{X}}(U)$.

Consider now the orbit $G.U_X$, its sum $W = \bigcup G.U_X$ (it is an open subset of $\overline{X}$) and its closure $\overline{W}$. We claim that $W = \overline{W}$. Indeed, if $y \in \overline{W} \setminus W$, then there is $U' = G \times U'_{\overline{X}} \in \mathcal{U}$ such that $y \in U'_{\overline{X}}$, but then $U'_{\overline{X}}$ intersects W, which means that U' intersects some gU, contradicting $\dim \mathcal{U} = 0$.

So W is a compact subset of $\overline{X}$ covered by the disjoint family $G.U_X$, meaning that the family must be finite. Thus, the orbit of U_X has k elements or – equivalently – the orbit of $G.U$ has k elements. Summing up, the stabiliser of U is a virtually cyclic group which is of index k in G, meaning that also G is virtually cyclic. □

Remark 4.2.2. *The above theorem*[2] *is also true when zero-dimensional coverings of* $G \times \partial X$ *are considered instead of zero-dimensional coverings of* $G \times \overline{X}$.

The remark explains why in the proof of 4.2.1 we did not rely on a (realistic) assumption that $\overline{X}$ is connected, which would make the proof easier. ∂X does not have to be connected as in the case of $G = \mathbb{F}_2$ (recall: 3.3.1), where ∂X is even zero-dimensional as a topological space.

[2] Including the proof.

Chapter 5

Different characterisations of equivariant asymptotic dimension

Asymptotic dimension has many equivalent characterisations. The following theorem ([6,15]) enumerates some of them.

Theorem 5.0.3. *Let X be a metric space. The following conditions are equivalent.*

1) $\operatorname{asdim} X \leq n$*;*

2) *for every $d < \infty$ there exists a uniformly bounded cover $\mathcal{V}$ of X with d-multiplicity*[1] *at most $n + 1$;*

3) *for every $\varepsilon > 0$ there is a uniformly cobounded*[2]*, ε-Lipschitz map $\varphi\colon X \to K$ to a simplicial complex of dimension n.*

4) *for every $r < \infty$ there exist uniformly bounded, r-disjoint*[3] *families $\mathcal{U}^0, \ldots, \mathcal{U}^n$ of subsets of X such that $\bigcup_i \mathcal{U}^i$ is a cover of X;*

We immediately observed (1.3.1) the similarity between the definitions of equivariant asymptotic dimension and asymptotic dimension (vide point 1) of the above theorem). Additionally, in 2.2 and 2.3 this correspondence was closer examined.

There exists an equivariant version of point 2) and equivalence of the definition of asdim and point 2) remains true in the equivariant case, as we will show in the next section.

Condition from point 3) corresponds closely to yet another characterisation of asdim in terms of property A. We will elaborate on that and provide an equivariant analogue in 5.2.

Although in Section 5.3 we will find a condition similar to point 3), the analogue of map φ will be ε-Lipschitz with respect to the G-coordinate, but not continuous with respect to the X-coordinate. An available weak notion of continuity will be sufficient to prove equivalence with $\text{eq-asdim}\, G \leq n$, but not enough to derive an analogue of point 4) in the same way as it is done for asdim. This leads to Question 5.3.6 asking if an analogue of point 4) is equivalent to the other conditions.

Comment 5.0.4. In all results of this chapter the compactification $\overline{X}$ can be replaced by its boundary ∂X. The proofs remain the same.

[1] See 5.1.1.

[2] There is a uniform bound on diameter of an inverse image of a simplex.

[3] r-disjointness of a family $\mathcal{U}_i$ means that distinct $U, U' \in \mathcal{U}_i$ are at least r-distant from each other.

5.1. d-multiplicity

Definition 5.1.1. For a given family $\mathcal{U}$ of subsets of a given metric space Y its d-multiplicity is the maximal number of the subsets intersecting a d-ball:

$$\max_{y \in Y} \#\{U \in \mathcal{U} \mid y^d \cap U \neq \emptyset\}.$$

Similarly, for a family $\mathcal{V}$ of subsets of $Y \times Z$ – where Z is any set – its Y-d-multiplicity is

$$\max_{(y,z) \in Y \times Z} \#\{V \in \mathcal{V} \mid y^d \times \{z\} \cap V \neq \emptyset\}.$$

Definition 5.1.2. A d-$\mathcal{VC}$yc-cover $\mathcal{V}$ is a $\mathcal{VC}$yc-cover such that any G-d-ball $g^d \times \{x\}$ intersects at most one element $V \in \mathcal{V}$ from each orbit of the G-action on $\mathcal{V}$.

Proposition 5.1.3. *For a pair $G, \overline{X}$ (similarly for $G, \partial X$), $n \in \mathbb{N}$ and $\alpha < \infty$ the following conditions are equivalent:*

*1) there exists an α-*eq-asdim-*covering of dimension at most n,*

2) there exists an open α-$\mathcal{VC}$yc-cover of G-α-multiplicity at most $n+1$.

Proof. 1) $\implies$ 2). Consider covering $\mathcal{U}$, which is a α-eq-asdim-covering for $G \times \overline{X}$ of dimension at most n. For $U \in \mathcal{U}$ set[4] $V(U) = \{(g, x) \in U \mid g^\alpha \times \{x\} \subseteq U\}$. By the definition of α-eq-asdim-covering $\mathcal{V} = \{V(U) \mid U \in \mathcal{U}\}$ is a covering of $G \times \overline{X}$. It is open: $(g, x) \in V(U)$ implies $g^\alpha \times \{x\} \subseteq U$, but U is open and g^α is finite, so it follows that $g^\alpha \times W \subseteq U$ – where W is a neighbourhood of x – and finally $\{g\} \times W \subseteq V(U)$. Covering $\mathcal{V}$ is clearly G-invariant.

We need to prove that $\mathcal{V}$ is a $\mathcal{VC}$yc-cover, so it is still to be checked, if $\mathcal{V}$ consists of $\mathcal{VC}$yc-subsets. Note that $V(U) \subseteq U$, so $gU \neq U$ equivalent to $gU \cap U = \emptyset$ implies that $gV(U) \cap V(U) = \emptyset$. On the other hand, $gU = U$ implies that $gV(U) = V(gU) = V(U)$. Thus $V(U)$ is a $\mathcal{VC}$yc-subset with the same stabiliser as U.

Now consider any G-α-ball: $g^\alpha \times \{x\}$. If it intersects $V(U)$ at (h, x), then clearly $h^\alpha \times \{x\}$ contains (g, x). But $h^\alpha \times \{x\} \subseteq U$ from the definition of $V(U)$, meaning that also U contains (g, x). Thus: firstly, the number of sets $V(U)$ intersecting $g^\alpha \times \{x\}$ does not exceed the number of sets U containing (g, x), which is bounded by $n+1$. Secondly, if $g^\alpha \times \{x\}$ intersects $V(U)$ and $gV(U)$, then U and gU intersects at (g, x), thus $U = gU$ and $V(U) = gV(U)$, so $\mathcal{V}$ is indeed a α-$\mathcal{VC}$yc-cover.

2) $\implies$ 1). Fix $\alpha < \infty$ and let $\mathcal{V}$ be an open α-$\mathcal{VC}$yc-cover of G-α-multiplicity at most $n+1$. Given $V \in \mathcal{V}$, we define $U(V) = V^\alpha$. The set $\mathcal{U}$ of such $\mathcal{U}(V)$ is clearly G-invariant, open and have G-Lebesgue number equal to α.

The stabiliser of V stabilises $U(V)$. Consider h that does not stabilise V. Then we know that there is no G-ball $g^\alpha \times \{x\}$ intersecting V and hV, which is equivalent to disjointness V^α and hV^α. Summarising, $U(V)$ is a $\mathcal{VC}$yc-subset with stabiliser equal to the stabiliser of V and $\mathcal{U}$ is a $\mathcal{VC}$yc-cover.

Similarly, $U(V)$ contains (g, x) if and only if $g^\alpha \times \{x\}$ intersects V. Thus multiplicity of $\mathcal{U}$ is bounded by G-α-multiplicity of $\mathcal{V}$, so we obtained $\dim \mathcal{U} \leq n$. □

[4] $V(U)$ is a kind of interior, but we do not select points with some neighbourhood in U, but with a "G-α-ball" in U.

5.2. Property A and equivariant asymptotic dimension

Let us recall the definition of property A of Guoliang Yu [11] reworded in language of ℓ_1-norms.

Definition 5.2.1. A discrete metric space (X, d) has property A if for all $\varepsilon > 0$, $R < \infty$, there is a family $\{A_x\}_{x\in X}$ of functions $A_x\colon X \to \mathbb{N}$ such that $0 < \|A_x\|_{\ell_1} < \infty$ for all $x \in X$ and:

- for all $x, y \in X$ such that $d(x, y) \le R$ we have $\frac{\|A_x - A_y\|}{\|\min(A_x, A_y)\|} < \varepsilon$
- there exists $S < \infty$ such that $\operatorname{supp} A_x \subseteq x^S$ for all $x \in X$.

In [8] Cencelj-Dydak-Vavpetič describe asymptotic dimension in terms similar to property A – they additionally require $\#\operatorname{supp} A_x \le n + 1$, where n is the asymptotic dimension of X. In particular, it shows that spaces of finite asymptotic dimension have property A.

A similar characterisation of equivariant asymptotic dimension was known to the experts – the supervisor of this thesis, Piotr Nowak was told about existence of such a characterisation by Arthur Bartels, who knew about it from Guoliang Yu and Rufus Willett. We will give a precise formulation in this section.

Before the proof, we need an additional lemma – it is trivial if we assume that Y is contractible, but since we allow (see 5.0.4) not only $Y = \overline{X}$[5] but also $Y = \partial X$, we give a proof in the general case.

Lemma 5.2.2. *Let $\alpha, n \in \mathbb{N}$, Y be a G-space and $\alpha, \#G \ge n+1$. Let $\mathcal{U}_0$ be an n-dimensional α-*eq-asdim*-covering of $G \times Y$. Then there exists an α-$\mathcal{VC}yc$-subcovering $\mathcal{U} \subseteq \mathcal{U}_0$ together with a system of distinct representatives; i.e., an injective function $c\colon \mathcal{U} \to G \times Y$ such that $c(U) \in U$ for all $U \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be the set of these $U \in \mathcal{U}_0$ that contain at least one set of the form $g^\alpha \times \{x\}$. It is a covering of $G \times Y$, because G-Lebesgue number of $\mathcal{U}_0$ is α and it is a $\mathcal{VC}$yc-cover, since $\mathcal{U}_0$ is such and our selection is G-invariant.

By induction over k (assuming that $\#G \ge k$) one can easily see that the open ball g^k has at least k elements and thus g^α has at least $n + 1$ elements. Consequently, with every $U \in \mathcal{U}$ we can associate its subset F_U with $n + 1$ elements. Thus, a sum of m of the sets F_U has at least m elements – each F_U has $n + 1$ elements, there are m of the sets F_U and each point is in at most $n + 1$ of the sets F_U as they are subsets of $U \in \mathcal{U}$. The infinite Hall's theorem assures existence of a system of distinct representatives for the family $(F_U \mid U \in \mathcal{U})$, which induces a system of distinct representatives for $\mathcal{U}$. □

Definition 5.2.3. A family $\{A_{g,x}\}_{(g,x)\in G\times\overline{X}}$ of functions $A_{g,x}\colon G \times \overline{X} \to \mathbb{N}$ is an $\mathcal{F}$-A-family if for every $(g_0, x_0) \in G \times \overline{X}$ the set

$$S_{g_0,x_0} = \{(g, x) \mid A_{g,x}(g_0, x_0) > 0\}$$

is open and the family $\mathcal{S}(A) = \{S_{g_0,x_0} \mid (g_0, x_0) \in G \times \overline{X},\ S_{g_0,x_0} \neq \emptyset\}$ is an $\mathcal{F}$-cover.

Definition 5.2.4. A family $\{A_{g,x}\}$ as above is semicontinuous if it is continuous as a function $A\colon G \times \overline{X} \to \mathbb{N}^{G\times\overline{X}}$, where $\mathbb{N}$ is equipped with the topology $\tau_+ = \{[n, \infty) \cap \mathbb{N} \mid n \in \mathbb{N} \cup \{\infty\}\}$ and $\mathbb{N}^{G\times\overline{X}}$ is considered with the product topology.

[5] In applications ([4]), $\overline{X}$ is required to be contractible.

The above definition is motivated by the fact that it is equivalent to the property that $A\colon (G\times \overline{X})\times(G\times\overline{X})\to\mathbb{N}$ is lower semicontinuous[6] with the second argument fixed; i.e., when $(g,x)\mapsto A_{g,x}(g_0,x_0)$ is lower semicontinuous.

Comment 5.2.5. In practise we will assume that the support of $A_{g,x}$ is finite, and then we can view the family $\{A_{g,x}\}$ as a function $A\colon G\times\overline{X}\to\ell_1(G\times\overline{X})$.

Theorem 5.2.6. *The following conditions are equivalent:*

1. eq-asdim $G\le n$ *(the compact set from the definition of* eq-asdim *is* $\overline{X}$*);*
2. *for every* $\varepsilon>0, R<\infty$ *there is a semicontinuous* $\mathcal{VC}yc$*-*A*-family* $\{A_{g,x}\}_{(g,x)\in G\times\overline{X}}$ *such that:* $\#(G\backslash\mathcal{S}(A))<\infty$*,* $\operatorname{supp} A_{g,x}\le n+1$ *and for* $d(g,h)\le R$ *we have:*
$$\frac{\|A_{g,x}-A_{h,x}\|_{\ell_1}}{\|\min(A_{g,x},A_{h,x})\|_{\ell_1}}\le\varepsilon;$$
3. *for every* $R<\infty$ *there is a* $\mathcal{VC}yc$*-*A*-family* $\{A_{g,x}\}_{(g,x)\in G\times\overline{X}}$ *such that:* $\operatorname{supp} A_{g,x}\le n+1$ *and for* $d(g,h)\le R$ *we have:*
$$\frac{\|A_{g,x}-A_{h,x}\|_{\ell_1}}{\|\min(A_{g,x},A_{h,x})\|_{\ell_1}}\le\frac{1}{n+1}.$$

Proof. $1\implies 2$. Let us fix $\varepsilon>0, R<\infty$. Choose an eq-asdim-covering $\mathcal{U}$ for $\alpha=2R+\frac{(2n+2)R}{\varepsilon}$ of dimension at most n. For every $U\in\mathcal{U}$ pick a distinct element $(g_U,x_U)\in U$ (without loss of generality $\#G=\infty$, $\alpha\ge\frac{n+2}{2}$ and we can use Lemma 5.2.2). Let:
$$l_U(g,x)=\left\lfloor\frac{\min\left(\alpha,\ \sup\{r\in[0,\infty)\mid\{g\}^r\times\{x\}\subseteq U\}\right)}{R}\right\rfloor$$
measure the "G-distance" from (g,x) to the exterior of U. We set:
$$A_{g,x}=\sum_{U\ni(g,x)}l_U(g,x)\cdot\mathbb{1}_{(g_U,x_U)},$$
where $\mathbb{1}_{(g_U,x_U)}$ denotes the characteristic function of the point (g_U,x_U). Note that for $d(g,h)\le R$ we have $|l_U(g,x)-l_U(h,x)|\le 1$ and because the dimension of $\mathcal{U}$ is at most n, there are at most $2(n+1)$ sets $U\in\mathcal{U}$ such that $l_U>0$ for (g,x) or (h,x). Consequently, $\|A_{g,x}-A_{h,x}\|\le 2n+2$ and since the G-Lebesgue number of $\mathcal{U}$ is at least α, we have $\|\min(A_{g,x},A_{h,x})\|\ge\frac{\alpha-R}{R}-1$. Finally: $\frac{\|A_{g,x}-A_{h,x}\|}{\|\min(A_{g,x},A_{h,x})\|}\le\frac{2n+2}{\frac{\alpha-2R}{R}}=\frac{(2n+2)R}{\frac{(2n+2)R}{\varepsilon}}=\varepsilon$.

Clearly $\operatorname{supp}A_{g,x}\le n+1$ as there are at most $n+1$ elements in the sum defining $A_{g,x}$. Let us notice[7] that $S_{g_U,x_U}\subseteq U$ – more precisely it consists of such points (g,x) that $g^R\times\{x\}$ is contained in U. So, if $hU\cap U=\emptyset$, then also $hS_{g_U,x_U}\cap S_{g_U,x_U}=\emptyset$. On the other hand if $hU=U$, we have:
$$(g,x)\in S_{g_U,x_U}\iff g^R\times\{x\}\subseteq U\iff h^{-1}g^R\times\{h^{-1}x\}\subseteq U\iff$$
$$(h^{-1}g,h^{-1}x)\in S_{g_U,x_U}\iff(g,x)\in hS_{g_U,x_U},$$

[6]Lower semicontinuity of a real function means that inverse images of open halflines (a,∞) are open. If the domain X is a metric space, it is equivalent to the fact that $\liminf_{x\to x_0}f(x)\ge f(x_0)$ for all $x_0\in X$.

[7]Here we use the fact that U is uniquely determined by (g_U,x_U).

thanks to the left-invariance of the group metric. Thus we conclude that $S_{g_U,x_U} \neq \emptyset$ has precisely the same stabiliser as U and thus it is a $\mathcal{VC}$yc-subset. More generally we can notice the equality $hS_{g_U,x_U} = S_{g_{hU},x_{hU}}$, which implies that $\#G\backslash\mathcal{S}(A) < \infty$ (by finiteness of $G\backslash\mathcal{U}$).

The last thing to show is the semicontinuity of A, because the openness of $\mathcal{S}(A)$ follows from semicontinuity. We will test continuity of $A\colon G \times \overline{X} \to \mathbb{N}^{G\times\overline{X}}$ on the following subbasis of $\mathbb{N}^{G\times\overline{X}}$:

$$\{f \in \mathbb{N}^{G\times\overline{X}} \mid f(g_0, x_0) \geq n\}_{n\in\mathbb{N}^+,\ (g_0,x_0)\in G\times\overline{X}}.$$

Note that we can restrict ourselves to (g_0, x_0) of the form (g_U, x_U) and $n > 0$.

Let us fix a set of the above form for $n > 0$ and some (g_U, x_U) and denote its inverse image by $S^n_{g_U,x_U}$. Fix any $(g, x) \in S^n_{g_U,x_U}$. We have: $g^{nR} \times \{x\} \subseteq U$. Equivalently, $x \in \bigcap_{h\in g^{nR}} U_h$, where for $U \in \mathcal{U}$ the set U_h is defined as the vertical slice $\{x \in \overline{X} \mid (h, x) \in U\}$. The product of this intersection and $\{g\}$, namely $\{g\} \times \bigcap_{h\in g^{nR}} U_h$, is contained in $S^n_{g_U,x_U}$ and is a neighbourhood of (g, x) as a product of an open set and a finite intersection of neighbourhoods.

$2 \implies 3$. Condition 3 is a special case of condition 2.

$3 \implies 1$. For $\alpha < \infty$ let $\{A_x\}$ be the functions with at most $n+1$ elements in the support satisfying $\frac{\|A_{g,x}-Ah,x\|}{\|\min(A_{g,x},A_{h,x})\|} \leq \frac{1}{n+1}$ for $d(g, h) \leq \alpha$. We define $\mathcal{U} = \mathcal{S}(A)$. Clearly the dimension of $\mathcal{U}$ is at most n and by assumption it is an open $\mathcal{VC}$yc-cover. Let us check that it has G-Lebesgue number α. For $(g, x) \in G \times \overline{X}$ let (g_0, x_0) maximise the function $A_{g,x}$. Clearly $(g, x) \in S_{g_0,x_0}$ and we have $A_{g,x}(g_0, x_0) \geq \frac{\|A_{g,x}\|}{n+1}$. If $(h, x) \notin S_{g_0,x_0}$, then $A_{h,x}(g_0, x_0) = 0$, so $\|A_{g,x} - A_{h,x}\| \geq A_{g,x}(g_0, x_0) \geq \frac{\|A_{g,x}\|}{n+1}$, in particular $\frac{\|A_{g,x}-A_{h,x}\|}{\|\min(A_{g,x},A_{h,x})\|} > \frac{1}{n+1}$, which implies $d(h, g) > \alpha$. Thus $g^\alpha \times \{x\} \subseteq S_{g_0,x_0}$. Finally, by the Proposition 1.2.8, we can choose a subcover $\mathcal{U}'$ such that $G\backslash\mathcal{U}'$ is finite. □

An significant drawback of the above characterisation is that the definition of a $\mathcal{VC}$yc-A-family is formulated in terms of the family $\mathcal{S}(A)$ and in the proof of implication $3 \implies 1$ the eq-asdim-covering $\mathcal{U}$ is defined as $\mathcal{S}(A)$ – the characterisation does not differ very much from the original. This seems inevitable as we can not require the family $A_{g,x}$ to be G-equivariant (by equivariance we mean that $A_{hg,hx}(g_0, x_0) = (hA_{g,x})(g_0, x_0) := A_{g,x}(h^{-1}g_0, h^{-1}x_0)$). That is because we allow the set $U \in \mathcal{U}$ to have the stabiliser in $\mathcal{F}$ – in particular with more than $n + 1$ elements – and the point (g_U, x_U) (see the proof of implication $1 \implies 2$) is moved by every $h \in G$. Then (a multiple of) the characteristic function $\mathbb{1}_{(g_U,x_U)}$ is a summand in $A_{hg,hx}$ for all h stabilising U, and – by equivariance – the functions $\mathbb{1}_{(h^{-1}g_U,h^{-1}x_U)}$ should be summands in $A_{g,x}$, contradicting the fact that the support of $A_{g,x}$ has at most $n+1$ elements.

We could cope this difficulty by arranging the domain of $A_{g,s}$ as not the set $G \times X$ itself, but its power set. Then we could "glue" all the characteristic functions $\mathbb{1}_{(h^{-1}g_U,h^{-1}x_U)}$ into one characteristic function, say of U. However, this approach is quite far away from the original definition of the property A, and that is why we decided to formulate the proposition in its current form first.

In the next section we implement the approach just described and alter the functions $A_{g,x}$ to lie on the ℓ_1-sphere (we no longer require natural values).

5.3. Equivariant asymptotic dimension via ε-maps to simplicial complexes or ℓ_p-spheres

Theorem 5.3.4 reformulates the problem using $\mathcal{F}$-A-families such that $A_{g,x} \in \ell_1(Y)$, where Y is some G-space. The G-space can be chosen to be a $\mathcal{VC}$yc-cover.

Definition 5.3.1. A G-set Y is called an $\mathcal{F}$-G-set if the stabilisers of $y \in Y$ are elements of $\mathcal{F}$.

Definition 5.3.2. Let Y be a $\mathcal{F}$-G-set. A family $\{B_{g,x}\}_{(g,x)\in G\times\overline{X}}$ of functions $B_{g,x}\colon Y \to [0,\infty)$ is an $\mathcal{F}$-B-family if:

- it is G-invariant; i.e., $(hB_{g,x})(y) := B_{g,x}(h^{-1}y) = B_{hg,hx}(y)$,
- for every $(g,x) \in G \times \overline{X}$, in $\operatorname{supp} B_{g,x}$ there is at most one element from each orbit of the G-action on Y.

Definition 5.3.3. We say that B as above is weakly semicontinuous if it is continuous as a function $G\times\overline{X} \to [0,\infty)^Y$, where the topology on $[0,\infty)$ is defined as $\tau_{\neq 0} = \{\emptyset, [0,\infty), (0,\infty)\}$.

Theorem 5.3.4. *The following conditions are equivalent:*

1. eq-asdim $G \leq n$ *(the compact set from the definition of* eq-asdim *is* $\overline{X}$*);*
2. *there is a countable $\mathcal{VC}yc$-G-space Y such that for every $\varepsilon > 0$ there is a weakly semicontinuous $\mathcal{VC}yc$-B-family $\{B_{g,x}\}_{(g,x)\in G\times\overline{X}}$ such that $\|B_{g,x}\|_{\ell_1(Y)} = 1$ and $\#\operatorname{supp} B_{g,x} \leq n+1$, which is ε-Lipschitz with respect to the variable g;*
3. *there is a countable $\mathcal{VC}yc$-G-space Y such that for every $\varepsilon > 0$ there is a weakly semicontinuous $\mathcal{VC}yc$-B-family $\{B_{g,x}\}_{(g,x)\in G\times\overline{X}}$ such that $\|B_{g,x}\|_{\ell_p(Y)} = 1$ and $\#\operatorname{supp} B_{g,x} \leq n+1$, which is ε-Lipschitz with respect to the variable g.*

Proof. $1 \implies 2$ Let $\mathcal{U}_k$ be the n-dimensional k-eq-asdim-coverings of $G\times\overline{X}$ and $Y = \bigcup_{k\in\mathbb{N}} \mathcal{U}_k$. It is clearly a countable $\mathcal{VC}$yc-G-set.

Let now $\varepsilon > 0$. We will proceed as in the proof of Theorem 5.2.6. Let $k = \left\lceil \frac{(2n+2)}{\varepsilon} \right\rceil$. For $U \in \mathcal{U}_k$ let

$$l_U(g,x) = \min\big(k,\ \sup\{r \in [0,\infty) \mid \{g\}^r \times \{x\} \subseteq U\}\big).$$

Note that this function is 1-Lipschitz with respect to the variable g and lower semicontinuous (the proof is the same as in 5.2.6).

We define $B^0_{g,x} = \sum_{U\ni x} l_U(g,x)\cdot \mathbb{1}_U$. Clearly $\#\operatorname{supp} B^0_{g,x} \leq n+1$. We also see that B is G-invariant and for $U \neq U'$ such that $U' = hU$, we have $B_{g,x}(U)B_{g,x}(U') = 0$, because U and U' are disjoint, so B is $\mathcal{VC}$yc-B-family. For $g,h \in G$ there are at most $2n+2$ sets $U \in \mathcal{U}$ such that $l_U > 0$ for (g,x) or (h,x), thus (by 1-Lipschitz property of l_U) $\|B^0_{g,x} - B^0_{h,x}\| \leq (2n+2)\cdot d(g,h)$.

Let now $B_{g,x} = \frac{B^0_{g,x}}{\|B^0_{g,x}\|}$. That is the moment, when we may loose semicontinuity (B^0 is semicontinuous, as l_U are lower semicontinuous), but weak semicontinuity is preserved, because $l_U(g,x) > 0$ if and only if $\frac{l_U(g,x)}{\|B^0_{g,x}\|} > 0$.

We have to check if the family is still Lipschitz with respect to the first argument. Let $g,h \in G$ and denote $d = d(g,h)$. Without loss of generality:

$$k \leq \|B^0_{g,x}\| \leq \|B^0_{h,x}\| \leq \|B^0_{g,x}\| + (2n+2)\cdot d,$$

thus we can write:

$$\left\| \frac{B^0_{g,x}}{\|B^0_{g,x}\|} - \frac{B^0_{h,x}}{\|B^0_{h,x}\|} \right\| \leq \left\| \frac{B^0_{g,x} - B^0_{h,x}}{\|B^0_{g,x}\|} + B^0_{h,x}\left(\frac{1}{\|B^0_{g,x}\|} - \frac{1}{\|B^0_{h,x}\|} \right) \right\| \leq$$

$$\frac{(2n+2)\cdot d}{k} + \left(\frac{\|B^0_{h,x}\|}{\|B^0_{g,x}\|} - 1 \right) \leq \frac{(2n+2)\cdot d}{k} + \frac{(2n+2)\cdot d}{k} < 2\varepsilon \cdot d.$$

$2 \implies 1$ Given $\alpha < \infty$, we choose $\varepsilon = \frac{1}{(n+1)\alpha}$ and a family $B_{g,x}$ for this ε. Let $S_y = \{(g,x) \in G \times \overline{X} \mid B_{g,x}(y) > 0\}$ and $\mathcal{U} = \{S_y \mid y \in Y\}$. By weak semicontinuity it is a family of open subsets and since each $B_{g,x}$ is nonzero, it is a covering. Clearly $\dim \mathcal{U} \leq n$.

Assume that $(g,x) \in S_y$; i.e. $B_{g,x}(y) > 0$. For $h \in G$ stabilising $y \in Y$ we have $B_{hg,hx}(y) = B_{g,x}(h^{-1}y) = B_{g,x}(y) > 0$. Moreover, if $hy \neq y$, we have $B_{hg,hx}(y) = B_{g,x}(h^{-1}y)$ and – by the second condition on a $\mathcal{VC}$yc-B-family – this value is equal to zero. Summarising, if $hy = y$, then $h(g,x) \in S_y$ and if $hy \neq y$, then $h(g,x) \notin S_y$, which asserts that S_y is a $\mathcal{VC}$yc-subset. We also have $hS_y = S_{hy} \in \mathcal{U}$, so $\mathcal{U}$ is a $\mathcal{VC}$yc-cover.

Let now $(g,x) \in G \times \overline{X}$ and y_0 be such an element $y \in Y$ that maximises $B_{g,x}(y)$. We have $B_{g,x}(y_0) \geq \frac{1}{n+1}$. Thus, since $h \mapsto B_{h,x}$ is ε-Lipschitz and $\varepsilon = \frac{1}{(n+1)\alpha}$, for any $h \in g^\alpha$, we have $B_{h,x}(y) \geq \frac{1}{n+1} - \frac{d(g,h)}{(n+1)\alpha} > 0$. Thus $g^\alpha \times \{x\} \subseteq S_{y_0} \in \mathcal{U}$. Finally, if $G\backslash\mathcal{U}$ is infinite, we choose a cofinite subcover using Proposition 1.2.8.

$2 \iff 3$. We will use the Mazur map f: for $2 \implies 3$ let $f((a_i)) = \left(a_i^{1/p}\right)$ and for $2 \impliedby 3$ let $f((a_i)) = (a_i^p)$, where $(a_i)_{i \in I} \in [0,\infty)^I$. Without loss of generality assume that we are proving 3 from 2.

Let $\varepsilon > 0$ and B be the family from point 2 that is δ-Lipschitz in variable g. The parameter $\delta > 0$ (depending only on ε) will be specified in a moment. Our candidate for a $\mathcal{VC}$yc-B-family is $B'_{g,x} = f(B_{g,x})$. Most of the conditions are obviously preserved – we only need to check Lipschitz constants. The Mazur map f is uniformly continuous[8], so if for $d(g,h) = 1$ we have $\|B_{g,x} - B_{h,x}\|_{\ell_1} \leq \delta$, then we also have $\|B'_{g,x} - B'_{h,x}\|_{\ell_p} \leq \varepsilon$ for an appropriately chosen δ for a given ε. As any $g,h \in G$ such that $d(g,h) = n$ can be connected by a chain $g = g_0, g_1, \ldots, g_n = h$ satisfying $d(g_i, g_{i+1}) = 1$, the above asserts that the map $(g,x) \mapsto B'_{g,x}$ is ε-Lipschitz with respect to the variable g. $\square$

Remark 5.3.5. *The condition "$\|B_{g,x}\|_{\ell_1(Y)} = 1$ and $\# \operatorname{supp} B_{g,x} \leq n+1$" in the above proposition is equivalent to saying that B acquires its values in an n-dimensional simplicial complex. Of course usage of the ℓ_1-norm on this complex (to define what an ε-Lipschitz map is) is arbitrary, as all norms are equivalent on finite dimensional vector spaces.*

Question 5.3.6. *Can we require the families $\mathcal{U} = \mathcal{U}(\alpha)$ guaranteeing that* eq-asdim $G \leq n$ *to be decomposable into (G-invariant) pairwise disjoint families $\{\mathcal{U}_i\}_{i=0}^n$?*

Comment 5.3.7. In the case of asymptotic dimension the transition from the family $\mathcal{U}$ with $\dim \mathcal{U} \leq n$ to pairwise disjoint families $\mathcal{U}_i$ can be done (([15]) by ε-maps to a simplicial complex similar to the maps $g \mapsto B_{g,x}$.

This method cannot be applied directly in our case, because the map $(g,x) \mapsto B_{g,x}$ is not continuous. It is not even semicontinuous – if for each (g_0,x_0) the map $(g,x) \mapsto B_{g,x}(g_0,x_0)$ is lower semicontinuous, then from the equality $\sum_{(g_0,x_0) \in G \times \overline{X}} B_{g,x}(g_0,x_0) = 1$ we can deduce that it is also upper semicontinuous, thus it is continuous. However, by the construction $B_{g,x}(g_0,x_0)$ acquires only rational values.

Note that in the proof of implication $2 \implies 1$ we do not use the fact that there is one Y for all ε or that it is countable. We also do not need the G-invariance of B – it is enough to have a weaker property: $\operatorname{supp} hB_{g,x} = \operatorname{supp} B_{hg,hx}$

Remark 5.3.8. *A direct construction of a family A described in condition 3 of Theorem 5.2.6 from a family B from 5.3.4 is also possible.*

[8] A reader unfamiliar with this general result can also use a compactness argument in our case.

Proof. Given $R < \infty$, we choose a family B for $\varepsilon = \frac{1}{3(n+2)^2 R}$. By multiplying each function $B_{g,x}$ by $S = 3(n+2)^2$, we get functions $SB_{g,x}$ of norm S and the family SB is $S\varepsilon$-Lipschitz in G-coordinate. Thus, for $d(g,h) \leq R$ we have $\|SB_{g,x} - SB_{h,x}\| \leq S\varepsilon R$ and $\|\min(SB_{g,x}, SB_{h,x})\| \geq S - S\varepsilon R$. Let $A_{g,x}(y) = \lceil SB_{g,x}(y) \rceil$. We get $\|A_{g,x} - A_{h,x}\| \leq S\varepsilon R + 2(n+1)$ and $\|\min(A_{g,x}, A_{h,x})\| \geq S - S\varepsilon R$. Notice that $S\varepsilon R = 1$. Now:

$$\frac{\|A_{g,x} - A_{h,x}\|}{\|\min(A_{g,x}, A_{h,x})\|} \leq \frac{1 + 2(n+1)}{3(n+2)^2 - 1} < \frac{2(n+2)}{2(n+2)^2} < \frac{1}{n+1}.$$

By any injection $Y \to G \times \overline{X}$ we can pull back the family $A_{g,x}$ from $\ell_1(Y)$ to $\ell_1(G \times \overline{X})$. Verification that the obtained family is indeed a $\mathcal{VC}$yc-A-family is easy to check. $\square$

Chapter 6

Transfer reducibility

We recall the notion of transfer reducible from [1]. The property is weaker than finiteness of eq-asdim (as we show in 6.0.16), but sufficient for proving the Farrell–Jones conjecture (see [2]).

We will start with a technical definition of an N-dominated space and go on to defining an equivalent of a group action of G on X.

Definition 6.0.9 (N-dominated space)**.** Let X be a metric space and $N \in \mathbb{N}$. The space X is controlled N-dominated if for every $\varepsilon > 0$, there is a finite CW-complex K of dimension at most N, and maps $i : X \to K$ and $p : K \to X$ and a homotopy H joining id_X and $p \circ i$ such that the diameter of the „trace" of x: $H(\{x\} \times [0,1])$ is at most ε for every $x \in X$.

Convention 6.0.10. In this chapter we *do not* assume that G acts on a space X.

Let S be a finite subset of a group G and X be a topological space.

Definition 6.0.11 (Homotopy S-action)**.** A homotopy S-action (ϕ, H) on X consists of continuous maps $\phi_g \colon X \to X$ for $g \in S$ and homotopies $H_{g,h} : X \times [0,1] \to X$ for $g, h \in S$ and $gh \in S$ joining $\phi_g \circ \phi_h$ with ϕ_{gh}. Additionally, if $1 \in S$, we require ϕ_1 to be the identity and $H_{1,1}$ to be a constant homotopy.

For a homotopy S-action (ϕ, H) by F_g we denote the set of all levels of homotopies $H_{h,i}$ ending at ϕ_g:

$$F_g = \{x \mapsto H_{h,i}(x,t) \mid hi = g,\ t \in [0,1]\}.$$

Definition 6.0.12. The $2n$-antidiagonal homotopic ball centred at $(g, x) \in G \times X$ (denoted by $ADB(g, x, n)$) is the set of all $(gh, z) \in G \times X$ such that there is:

1. a sequence $x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n$ with $x_0 = x$ and $x_n = z$;
2. a sequence $a_1, b_1, \ldots, a_n, b_n \in S$ such that $a_1^{-1} b_1 \cdots a_n^{-1} b_n = h$;
3. sequences $(f_i^a)_{i=1}^n, (f_i^b)_{i=1}^n$ such that $f_i^a \in F_{a_i}$ and $f_i^b \in F_{b_i}$ satisfying: $f_i^b(x_i) = y_{i-1} = f_{i-1}^a(x_{i-1})$.

The situation in the above definition can be described by the following diagram.

$$\begin{tikzcd}
x = x_0 \arrow[dr, "f_1^a"] & & \cdots \arrow[dl, "f_1^b"'] \arrow[dr, dotted] & & x_{n-1} \arrow[dl, "f_{n-1}^b"] \arrow[dr, "f_{n-1}^a"] & & x_n = z \arrow[dl, "f_n^b"] \\
& y_0 & & \cdots & & y_{n-1} &
\end{tikzcd}$$

If the action is not a homotopy action, but an ordinary G-action, then an equivalent of f_i^a would be a_i as an automorphism of X, and the above "ball" would be the set of points of the form $(gh, h^{-1}x)$ for h of the form $a_1^{-1}b_1 \cdots a_n^{-1}b_n$. If we think of S as a symmetric set of generators of G containing 1, then it is clear, why we called the above set a "$2n$-ball" and the presence of elements h, h^{-1} in the formula explains, why we called it "antidiagonal".

Definition 6.0.13 (α-long cover)**.** For $\alpha \in \mathbb{N}$ a cover $\mathcal{U}$ of $G \times X$ is α-long with respect to a homotopy S-action (ϕ, H), if for every $(g, x) \in G \times X$, there is $U \in \mathcal{U}$ containing $ADB(g, x, \alpha)$.

Definition 6.0.14 (Transfer reducible)**.** We say that a group G is transfer reducible over a family of subgroups $\mathcal{F}$ if there is a number N with the following property.

For every finite subset S of G there are:

i) a contractible compact controlled N-dominated metric space X,
ii) a homotopy S-action (ϕ, H) on X,
iii) a open cover $\mathcal{U}$ of $G \times X$,

satisfying:

a) $\dim\mathcal{U} \leq N$,
b) $\mathcal{U}$ is $\#S$-long with respect to (ϕ, H),
c) $\mathcal{U}$ is an open $\mathcal{F}$-cover with respect to the G action on $G \times X$ given by $g.(h, x) = (gh, x)$.

Sometimes, to emphasise the parameter N we will write that G is N-transfer reducible with respect to $\mathcal{F}$.

The first step in noticing the correspondence between eq-asdim and transfer reducibility is the following remark.

Remark 6.0.15. *Let G be finitely generated group G, $\mathcal{F}$ be a family of subgroups and $N \in \mathbb{N}$. Under the technical assumption that $\#G = \infty$ or $G \in \mathcal{F}$ the following conditions are equivalent:*

1. *for every finite $S \subseteq G$ and $\alpha \in \mathbb{N}$ there are X, (ϕ, H) and $\mathcal{U}$ as above such that:*
   (a) $\dim\mathcal{U} \leq N$,
   (b) $\mathcal{U}$ is α-long with respect to (ϕ, H),
   (c) $\mathcal{U}$ is an open $\mathcal{F}$-cover with respect to the G action on $G \times X$ given by $g.(h, x) = (gh, x)$;
2. *G is N-transfer reducible over $\mathcal{F}$;*
3. *for each $r \in \mathbb{N}$, $S = 1^r$ and each $\alpha \in \mathbb{N}$ there are X, (ϕ, H) and $\mathcal{U}$ as above such that $\mathcal{U}$ is α-long with respect to (ϕ, H) and a), c) as above;*
4. *for each $r \in \mathbb{N}$ and $S = 1^r$ there are X, (ϕ, H) and $\mathcal{U}$ as above such that $\mathcal{U}$ is $\#S$-long with respect to (ϕ, H) and a), c) as above.*

Proof. The implication $1 \implies 2$ is obvious as it is enough to fix $\alpha = \#S$. The same with the implication $1 \implies 3$, because we fix $S = 1^r$. Similarly for $2 \implies 4$ and $3 \implies 4$.

It suffices to prove $4 \implies 1$. If G is finite and thus $G \in \mathcal{F}$ we can take a trivial action on $X = \{x_0\}$ and $\mathcal{U} = \{G \times X\}$, so we will assume that G is infinite.

Let S be a finite subset as in point 1. We take a ball 1^r large enough to contain S and have at least α elements. From 4 we have X, a homotopy action and a covering $\mathcal{U}$, which has dimension bounded by N, is $\#1^r$-long for $\#1^r \geq \alpha$, and is an open $\mathcal{F}$-cover. By restricting the homotopy 1^r-action $((\phi)_{s\in 1^r}, (H_{g,h})_{g,h\in 1^r})$ to the S-action $((\phi)_{s\in S}, (H_{g,h})_{g,h\in S})$, we get the claim. □

The next proposition and corollary are based on [2, Proposition 2.1].

Proposition 6.0.16. *Assume that* $\text{eq-asdim}\, G \le N$ *with* $\overline{X}$ *being contractible and* N*-dominated. Then* G *is* N*-transfer reducible.*

Proof. Let S be a finite subset of G. The space X from the definition of transfer reducibility will be $\overline{X}$ from the assumptions.

As we noticed in comments after Definition 6.0.14, given a group action we can establish a homotopy action by: $\phi_g(x) = gx$, $H_{g,h}(t,x) = ghx$. Then $F_g = \{\phi_g\}$ and the homotopic antidiagonal balls are of the following form $ADB(g,x,n) = \{(gh, h^{-1}x) \mid h = a_1^{-1}b_1 \cdots a_n^{-1}b_n;\ a_i, b_i \in S\}$.

We fix α such that 1^α contains the set $\{a_1^{-1}b_1 \cdots a_{\#S}^{-1}b_{\#S} \mid a_i, b_i \in S\}$. By $\text{eq-asdim}\, G \le N$ we have a covering $\mathcal{U}_0$ of $G \times \overline{X}$ of dimension at most N, with G-Lebesgue number α and being a $\mathcal{VC}$yc-cover with respect to the diagonal G-action.

We pull this covering back by an automorphism of $G \times \overline{X}$, which is equivariant with respect to the diagonal action in the codomain and the action trivial on $\overline{X}$ in the domain: $(g,x) \overset{\psi}{\mapsto} (g, gx)$. The new covering is $\mathcal{U} = \{\psi^{-1}(U_0) \mid U_0 \in \mathcal{U}_0\}$.

$\mathcal{U}$ is obviously an open $\mathcal{VC}$yc-cover (with respect to the action $g(h,x) = (gh,x)$) with dimension bounded by N. We can also notice that it is $\#S$-long, because of the fact that

$$ADB(g,x,\#S) = \{(gh, h^{-1}x) \mid h = a_1^{-1}b_1 \cdots a_{\#S}^{-1}b_{\#S}\} \subseteq \{(gh, h^{-1}x) \mid h \in 1^\alpha\}$$

and the following equivalence:

$$\begin{gathered}\{(i, gx) \mid i \in g^\alpha\} = \{(gh, gx) \mid h \in 1^\alpha\} = \\ \psi[\{(gh, h^{-1}x) \mid h \in 1^\alpha\}] \subseteq \psi[U] \\ \Longleftrightarrow \\ \{(gh, h^{-1}x) \mid h \in 1^\alpha\} \subseteq U.\end{gathered}$$

□

So we can see that transfer reducibility is indeed a property weaker than finite eq-asdim if we omit the technical assumptions about X for the first notion and lack of such assumptions for the second. As we can see from the proof, the fact that group actions on the Cartesian product differ in the two definitions is unimportant – we can rewrite the definition of eq-asdim so that these actions agree. The important difference is the fact that we can choose different spaces X in the definition of transfer reducible and we do not require an actual group action, but accept a homotopy action.

As explained in quotation 1.3.2 in practise (see [2]) spaces X are chosen to be big balls in a space Y equipped with a G-action. The group-action homeomorphism $\phi_g^0(y) = gy$ is restricted to that ball and composed with a retraction of Y onto X to produce ϕ_g and then the necessary homotopies $H_{g,h}$ are constructed using the fact that Y is a uniquely geodesic space.

Corollary 6.0.17. *Hyperbolic groups are transfer reducible over* $\mathcal{VC}$*yc.*

Sketch of proof. The fact that hyperbolic groups have finite equivariant asymptotic dimension is the main result of [3]. The space X used to show that fact is a so-called Rips complex of G (with sufficiently big parameter) which (in this case) is a finite dimensional contractible simplicial complex and has a compactification $\overline{X}$ being contractible and metrisable[1].

[1] We refer the reader to [2, Proposition 2.1].

Moreover, there is a homotopy such that $H(\,\cdot\,,0) = id_{\overline{X}}$ and for all $t > 0$: $H(\overline{X}, t) \subseteq X$. So – by compactness – the function $x \mapsto H^t(x) := H(x,t)$ has image contained in a finite subcomplex K_t of X. Again by compactness for each $\varepsilon > 0$ there is $\delta > 0$ such that $H_\delta(x,t) = H(x, \delta t)$ satisfies $\operatorname{diam} H_\delta(\{x\}, [0,1]) < \varepsilon$. Summing up, $p = H^\delta$ and the inclusion $i\colon K_\delta \to \overline{X}$ together with a homotopy H_δ joining $id_{\overline{X}}$ and $i \circ p$ guarantee that $\overline{X}$ is N-dominated, where N is the dimension of X as a simplicial complex.

Finally, Proposition 6.0.16 can be applied. □

Chapter 7

Equivariant topological dimension

Recall that the Lebesgue covering dimension of a topological space X is the smallest integer n such that any open covering has a refinement[1] of dimension at most n. The number n is sometimes called *the* topological dimension and is denoted by $\dim X$.

If X is an F-space for some group F, a natural question to ask is whether any F-covering has an F-refinement of dimension n. By an F-covering we mean an $\mathcal{F}$-cover, where $\mathcal{F}$ is the family of all subgroups of F. In other words: the covering is F-invariant and two distinct elements of an orbit are disjoint.

The question was asked and answered in positive in [10] for a finite group F acting on a metric space by isometries. This made the bound in propositions 3.2, 3.3 of [3] independent of the order of the group F.

In [3, 5] a bound on the orders of the finite subgroups F of a group was needed. Due to the above improvement, in [16] a proof of the Farrell–Jones conjecture became possible in a situation, where no such bound exists.

We will prove that the assumption that the group F acting on the space is finite, is superfluous. It is enough to assume properness of the action.

7.1. Dimension theory – necessary definitions and facts

Recall some definitions and facts from dimension theory after [13]. In brackets, we reference the corresponding statements from [13].

Definition 7.1.1 (5.1.1)**.** The local dimension, $\operatorname{loc}\dim X$, of a topological space X is defined as follows. If X is empty, then $\operatorname{loc}\dim X = -1$. Otherwise, $\operatorname{loc}\dim X$ is the smallest integer n such that for every point $x \in X$ there is an open set $U \ni x$ such that $\dim \overline{U} \leq n$. If there is no such n, then $\operatorname{loc}\dim X = \infty$.

Theorem 7.1.2 (5.3.4)**.** *If X is a metric space, then* $\operatorname{loc}\dim X = \dim X$.

Corollary 7.1.3. *If V is an open subset of a metric space X, then* $\dim V \leq \dim X$.

Proof. It is enough to prove the claim for $\operatorname{loc}\dim$. Consider $x \in V$. There is an open (in X) set U_0 with $\dim \overline{U}_0 \leq \operatorname{loc}\dim X$. We also have an open neighbourhood $V_x \ni x$ such that $\overline{V}_x \subseteq V$. Thus $U = U_0 \cap V_x$ is an open neighbourhood of x, its closure in X is equal to its closure in V and it is a closed subset of $\overline{U}_0$, so $\dim \overline{U} \leq \dim \overline{U}_0 \leq \operatorname{loc}\dim X$ (dimension of a closed subset never exceeds dimension of the space). Thus $\operatorname{loc}\dim V = \sup_{x \in V} \inf_{U \ni x} \dim \overline{U} \leq \operatorname{loc}\dim X$, as needed (where U are open sets of V and closures are taken in V). $\square$

[1]Family of sets $\mathcal{V}$ refines $\mathcal{U}$ if any $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.

Corollary 7.1.4. *In the case of metric spaces, there is no need for taking closures of neighbourhoods in the definition of local dimension (it is enough to consider open neighbourhoods and calculate their dimension).*

Proof. Fix $x \in X$. It suffices to check the equality $\inf_{U \ni x} \dim \overline{U} = \inf_{U \ni x} \dim U$, where U are open neighbourhoods of x. Let U_x be an open neighbourhood of x such that $\overline{U}_x$ has the smallest possible dimension. Then dimension of U_x – by Corollary 7.1.3 – is no larger. On the other hand, if there is an open neighbourhood V of x such that $\dim V < \dim \overline{U}_x$, then there would be an open neighbourhood W such that $\overline{W} \subseteq V$ and thus $\dim \overline{W} \leq \dim V < \dim \overline{U}_x$ contradicting the minimality of U. □

Proposition 7.1.5. *The dimension of a metric space X is equal to the supremum of dimensions of its open subsets. It is enough to consider a supremum over any open cover of X.*

Proof. Let $\mathcal{U}$ by any open covering of X. By Corollary 7.1.3, the dimension of X is no smaller than dimensions of its open subsets, thus $\dim X \geq \sup_{U \in \mathcal{U}} \dim U$. On the other hand, it is equal to the local dimension, which is equal – by Corollary 7.1.4 – to the supremum over points of infima over open neighbourhoods of their dimensions. But clearly, we have the inequality:

$$\dim X = \sup_x \inf_{U \ni x} \dim U \leq \sup_x \inf_{\mathcal{U} \ni U \ni x} \dim U \leq \sup_{x,\, \mathcal{U} \ni U \ni x} \dim U = \sup_{U \in \mathcal{U}} \dim U. \quad \square$$

Theorem 7.1.6 (9.2.16)**.** *Let $f \colon X \to Y$ be a continuous open surjection of metrisable spaces. If every fibre $f^{-1}(y)$ is finite, then* $\dim X = \dim Y$.

7.2. Equivariant refinements

The following proposition strengthens [10, Corollary 2.5].

Proposition 7.2.1. *Let (Y, d) be a metric space with an isometric proper action of a group H. Then* $\dim H \backslash Y = \dim Y$.

Proof. We can fix a pseudometric on the quotient space: $d'([y],[y']) = \inf_{h,h' \in H} d(hy, h'y')$. The action is isometric, so it is equal to $\inf_{h \in H} d(hy, y')$. If $[y] \neq [y']$, then – by properness of the action – there is no infinite sequence $h_n y$ convergent to y' and thus $d'([y],[y']) > 0$. Thus $H \backslash Y$ is a metric space (it is easy to check, that the quotient topology and the metric topology agree).

Let $y \in Y$. Similarly as above, there is $\varepsilon > 0$ such that $y^{2\varepsilon}$ is disjoint with all the other elements of the orbit Hy. Consequently, y^{ε} is disjoint with its translates and has a finite stabiliser S (the one of y).

Denote by g the restriction of f to y^{ε}. For $y' \in y^{\varepsilon}$ and $z' = g(y')$, the fibre $g^{-1}(z')$ is contained in Sy' and thus finite. Clearly g is an open surjection onto its (open) image, so Theorem 7.1.6 applies: $\dim f(y^{\varepsilon}) = \dim g(y^{\varepsilon}) = \dim y^{\varepsilon}$.

Using the openness and the surjectivity again, we notice, that the family $\{f(y^{\varepsilon})\}$ – where $y \in Y$ and $\varepsilon = \varepsilon(y)$ – is an open covering of $H \backslash Y$. With Proposition 7.1.5 we conclude:

$$\dim H \backslash Y = \sup \dim f(y^{\varepsilon}) = \sup \dim y^{\varepsilon} = \dim Y. \quad \square$$

Finally, we can prove a version of [10, Proposition 2.6].

Proposition 7.2.2. *Let Y be a metric space with an isometric proper action of a group H and* $\dim Y = n$. *Any open H-cover $\mathcal{U}$ of Y has an open H-refinement $\mathcal{W}$ with dimension at most n.*

Proof. Denote the quotient map by q. By Proposition 7.2.1, we know that the open covering $\{q(U) \mid U \in \mathcal{U}\}$ of $H\backslash Y$ has a refinement $\mathcal{V}$ of dimension at most n.

Clearly $q^{-1}(V)$ for $V \in \mathcal{V}$ is H-invariant, in particular it is an H-subset. The covering $\{q^{-1}(V) \mid V \in \mathcal{V}\}$ has the same dimension as $\mathcal{V}$.

In order to obtain the required refinement of $\mathcal{U}$, it is enough to divide each $q^{-1}(V)$ into appropriate disjoint parts. Note that a division into disjoint parts does not increase the dimension of a covering. Let U_V be such an element of $\mathcal{U}$ that $V \subseteq q(U_V)$. Then clearly:

$$q^{-1}(V) \subseteq q^{-1}(q(U_V)) = \bigsqcup_{[h]\in H/S} hU_V,$$

where S is the stabiliser of U_V. The required division of $q^{-1}(V)$ is $\bigsqcup_{[h]} q^{-1}(V) \cap hU_V$. The covering $\mathcal{W} = \{q^{-1}(V) \cap hU_V \mid V \in \mathcal{V},\ h \in H\}$ is clearly an H-covering[2] and refines $\mathcal{U}$. $\square$

[2] Moreover, if $\mathcal{U}$ is an $\mathcal{F}$-cover, then $\mathcal{W}$ also is, as the stabiliser of $q^{-1}(V) \cap U_V$ is the same as the stabiliser of U_V.

Summary

In the light of our findings, the geometry of Cartesian product of a group and its appropriate boundary as a G-space is crucial for the notion of equivariant asymptotic dimension. It may suggest, that the boundary, not the compactification, is the appropriate space to occur in the definition of eq-asdim. Nonetheless, in practise, the whole compactification ([4]) or even parts of its "interior" ([1]) are used and different restrictions (contractibility, CW-structure) are put on them.

The study of coverings of products of a discrete group with a relatively simple topological space (like the Cantor set in the case of $\mathbb{F}_2$) appears feasible and has unexpectedly far reaching consequences for important topological conjectures of Farrell–Jones and Borel. However, finding such coverings is much more involved than one could initially suppose as we deduce from our considerations, the volume of article [3] and the lack of other results. In this context, the study of transfer reducibility – even if (or perhaps because of this fact) the formulation of the notion is more complicated – can appear more promising.

We provided a number of alternative characterisations of equivariant asymptotic dimension in language of d-multiplicity, maps into ℓ_p-spaces similar as for property A and ε-maps into simplicial complexes. It is an open question if an eq-asdim-covering of dimension n can be divided into $n+1$ disjoint families.

We proved a theorem relating a geometric property of equivariant asymptotic dimension with the algebraic property of being virtually cyclic. It asserts that eq-asdim equal to zero is equivalent to virtual cyclicity.

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