1. Let $A_1, \dots, A_m$ be distinct subsets of an n element set. Suppose that $A_i \cap A_j \neq \emptyset$ for all i, j. Show that $m \leq 2^{n-1}$.

   Solution. Let $\mathcal{F}$ be the family of $A_1, \dots, A_m$. Since $A \in \mathcal{F}$, $A^{\complement} \notin \mathcal{F}$, because $A \cap A^{\complement} = \emptyset$. There exist exactly 2^n distinct subsets of an n element set. Thus, $m \leq 2^{n-1}$, as each set in $\mathcal{F}$ excludes exactly one from the family of all subsets.

2. Prove that if $\mathcal{F}$ is a family of distinct pairwise intersecting subsets of an n element set X, then there exists a family $\mathcal{F}'$ of distinct pairwise intersecting subsets of X, such that $\mathcal{F} \subseteq \mathcal{F}'$ and $|\mathcal{F}'| = 2^{n-1}$.

   Solution. By contradiction: we assume that no set can be added to $\mathcal{F}$ and $m < 2^{n-1}$. Then, we can choose set A such that $A, A^{\complement} \notin \mathcal{F}$. As both A and $A^{\complement}$ cannot be added to $\mathcal{F}$, we can find sets B and C in $\mathcal{F}$, such that $A \cap B = \emptyset$ and $A^{\complement} \cap C = \emptyset$. It follows $B \subset A^{\complement}$ and $C \subset A$, so $B \cap C = \emptyset$. The contradiction proves $m = 2^{n-1}$.

3. Let $A_1, \dots, A_m$ be a family of distinct subsets of an n element set, such that $|A_i|$ and $|A_i \cap A_j|$ are even for all i, j. Prove that $m \leq 2^{[n/2]}$. Is this bound tight?

   Solution. Let $v_1, \dots, v_m \in \mathbb{Z}_2$ vectors such that jth coordinate of vector v_i is 1 iff $j \in A_i$. Consider the standard scalar product where the sum is taken in $\mathbb{Z}_2$. Let U be $\operatorname{span}\{v_1, \dots v_m\}$. By the assumptions we have $\langle v_i, v_j \rangle = 0$ for all i, j so $U \subseteq U^{\perp}$. As $\dim U + \dim U^{\perp} = n$, $\dim U \leq \left\lfloor \frac{n}{2} \right\rfloor$. We get $m \leq |U| = 2^{\dim U} \leq 2^{\left\lfloor \frac{n}{2} \right\rfloor}$
   Consider family $\mathcal{A}$ of subsets $\{1, 2\}, \{3, 4\}, \dots, \{2\left\lfloor \frac{n}{2} \right\rfloor - 1, 2\left\lfloor \frac{n}{2} \right\rfloor\}$. Family $\mathcal{F}$ of subsets such that every $A \in \mathcal{F}$ is an union of a number of sets from $\mathcal{A}$ (including the empty set) satisfies the assumptions and $|\mathcal{F}| = 2^{\left\lfloor \frac{n}{2} \right\rfloor}$. It proves the bound is tight.

4. Let n be odd. Let $A_1, \dots, A_m$ be a family of distinct subsets of an n element set, such that $|A_i|$ is even for all i and $|A_i \cap A_j|$ is odd for all i, j. Prove that $m \leq n$. Is this bound tight?

   Solution. We consider the complements of $A_1, \dots, A_m$ and get the thesis of Clubs in Oddtown Problem. Indeed, $|A_i^C| = n - |A_i| \equiv 1 - 0 = 1 (\operatorname{mod} 2)$ and $A_i^C \cap A_j^C = n - |A_i| - |A_j| + |A_i \cap A_j| \equiv 1 - 0 - 0 + 1 \equiv 0 (\operatorname{mod} 2)$. So, we get the inequality $m \leq n$.

5. Let A be a $2n \times 2n$ matrix with zeroes on the main diagonal and ± 1 elsewhere. Show that A is non-singular over $\mathbb{R}$.

   Solution. We will show that A is non-singular over $\mathbb{Z}_2$. Let $v_1, \dots, v_m$ be the rows of the matrix. Over $\mathbb{Z}_2$, we have $-1 = 1$ so $v_i = (1, 1, ..., 1, 0, 1, ..., 1)$ such that 0 is the ith coordinate of v_i. Consider again the standard scalar product where the sum is taken in $\mathbb{Z}_2$. $\langle v_i, v_i \rangle = 2n - 1 (\operatorname{mod} 2) = 1$ for all i and $\langle v_i, v_j \rangle = 2n - 2 (\operatorname{mod} 2) = 0$ for all $i \neq j$. It means that $v_1, \dots, v_m$ are orthogonal, so as a result they are linearly independent.

6. Suppose $\mathbb{F}$ is a subfield of $\mathbb{G}$. Suppose $v_1, \dots, v_k$ are linearly independent in the vector space $(\mathbb{F}^n, \mathbb{F})$. Does it follow that $v_1, \dots, v_k$ are linearly independent in the vector space $(\mathbb{G}^n, \mathbb{G})$?

   Solution. $v_1, \dots, v_k$ are linearly independent vectors in the vector space $(\mathbb{F}^n, \mathbb{F})$ so there exist vectors $v_{k+1}, \dots, v_n \in \mathbb{F}^n$ such that set $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{F}^n$. Take matrix A whose rows are vectors $\{v_1, \dots, v_n\}$. $\operatorname{rk}(A) = \dim(\operatorname{span}(v_1, \dots, v_n) = n$. It is maximal so $\det(A) \neq 0$ in the field $\mathbb{F}$ as well as in the field $\mathbb{G}$, because $\mathbb{F}$ is a subfield of $\mathbb{G}$. It proves that $v_1, \dots, v_n$ are linearly independent in the vector space $(\mathbb{G}^n, \mathbb{G})$.

7. A family $S_1, \dots, S_k$ of subsets of a given set X is called a *sunflower* with k petals and core A (it could be that $A = \emptyset$) if $S_i \cap S_j = A$ for all $i \neq j$ and $S_i \setminus A$ is nonempty for all i.

   Prove that every family of s element subsets of X satisfying $|\mathcal{F}| > s!(k-1)^s$ contains a sunflower with k petals.

   Solution. Induction by s.

8. Let $\mathcal{F}$ be an antichain of subsets (with an inclusion order) of an n element set. Suppose that all of these sets have cardinality at most k where $2k \leq n$. Show that $|\mathcal{F}| \leq \binom{n}{k}$.

9. Let $n \leq 2k$ and let $A_1, \dots, A_m$ be distinct k element subsets of a give set X with n elements. Suppose $A_i \cup A_j \neq X$ for all i, j. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$.

   Solution. By considering complements to sets $A_1, \dots, A_m$ we get the thesis of Erdos-Ko-Rado theorem.

10. Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be subsets of a given finite set X such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Let $a_i = |A_i|$ and $b_i = |B_i|$. Prove the inequality

$$\sum_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

11. Let $A_1, \dots, A_m$ be a element subsets and $B_1, \dots, B_m$ be b element subsets of a given finite set X, such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Show that $m \leq \binom{a+b}{a}$. Is this bound tight?

12. Let $v_1, \dots, v_n$ be real numbers such that $|v_i| \geq 1$ for $i = 1, \dots, n$. Define

$$A = \{x = (x_1, \dots, x_n) \in \{-1, 1\}^n : \quad |v_1 x_1 + \cdots + v_n x_n| < 1\}.$$

   Prove that $|A| \leq \binom{n}{[n/2]}$.

   In other words, the probability that the n step random walk with steps $\pm v_i$ (each taken with probability $\frac{1}{2}$) ends up in the interval $[-1, 1]$ is upper bounded by $2^{-n}\binom{n}{[n/2]} = O(1/\sqrt{n})$.

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   Prove that $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.
   In other words, the probability that the n step random walk with steps $\pm v_i$ (each taken with probability $\frac{1}{2}$) ends up in the interval $[-1, 1]$ is upper bounded by $\binom{2^{-n}n}{\lfloor n \rfloor n/2] = O(1/\sqrt{n})}$

   Solution. Without loss of generality, we can assume that $v_1, \dots, v_n > 0$. For every vector $x = (x_1, \dots, X_n) \in A$ consider subset X of an n element set $\{1, 2, \dots, n\}$ such that $i \in X$ iff $x_i = 1$. Subsets constructed in this operation form an antichain in $\{-1, 1\}^n$. Indeed, if $X \subsetneq Y$ there exists i such that $x_i = -1$ and $y_i = 1$. Then $|x_1 V - 1 + \ldots + x_n v_n| + |y_1 v_1 + \ldots + y_n v_n| \geq |(y_1 - x_1) v_1 + \ldots + (y_n - x_n) v_n| \geq 2$. As a result, either x or y does not belong to A. From Sperner's lemma we get $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

2. Suppose that in a given finite partial order the maximal length of a chain is equal to r. Prove that this partial order can be partitioned into r antichains.

   Proof. Let f be a function, such that $f(x)$ is length of the longest chain ended by x (i.e. length of the longest chain $x_1 \preceq x_2 \preceq ... \preceq x$). We will show thet for every $i \in \{1, 2, ..., \}$ set $A_i := \{x \in S : f(x) = i\}$ is antichain (obviously $\forall\ i \in \{1, 2, ..., r\} \quad A_i \neq \varnothing$). Assume by contradiction that $\exists\ x, y \in A_i :\ x \preceq y$. Then:

$$f(x) = i \Rightarrow \underbrace{x_1 \preceq x_2 \preceq ... \preceq x}_{i-\text{elements}} \preceq y \Rightarrow f(y) = i + 1$$

   Thus A_i is antichain for every $i \in \{1, 2, ..., r\}$. □

3. Let s, r be positive integers. Show that in any partial order on a set of $n \geq sr + 1$ elements, there exists a chain of length $s + 1$ or an antichain of size $r + 1$.

   Solution. By contradiction, assume that both: the length s' of the longest chain in this partial order P is at most s and the length r' of the biggest antichain is at most r. From Dilworth's theorem we get that P can be partitioned into r' chains. The cardinality of P is at most $r's' \leq rs < rs + 1$.

4. Let s, r be positive integers. Show that every sequence of $sr + 1$ real numbers contains a non-decreasing subsequence of length $s + 1$ or a non-increasing subsequence of length $r + 1$.

   Solution. This is the thesis of Erdős-Shekeres theorem.

5. Show that the above theorem is tight.

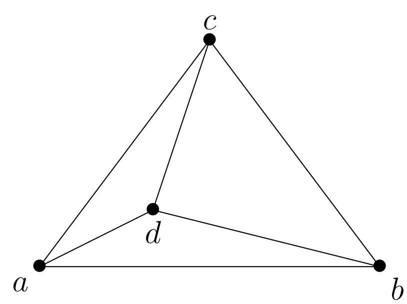
   Solution.

6. Find $R(3, 3)$ and $R(4, 3)$.

Solution. First, we will show that $R(3,3) \leq 6$. We consider a complete graph G with six vertices. From a vertex v from G there goes five edges to other vertices from G. By pigeonhole principle three of them are in the same color (suppose it is blue). As edges (v, v_1) , (v, v_2), (v, v_3) are blue, one of edges between vertices v_1, v_2, v_3 is blue or all of them are red. In both cases we get one-colored triangle.

7. Show that for any integer $m \geq 3$ there exists an integer $n = n(m)$ such that any set of n points in the Euclidean plane, no three of which are collinear, contains m points which are the vertices of a convex m-gon.

   Proof. Let $n = R_2(3, m)$ and A be set of n points in the Euclidean plane. Let us define coloring of triples of points, such that $\chi(a, b, c) = 1$, if number of points lying in the interior of triangle spanned by a, b, c is odd and $\chi(a, b, c) = 0$ otherwise. Then, from definition of n, there exists set $S \subseteq A$, such that $|S| = m$ and every triple of points contained in S has the same color. Then S is convex m-gon, because by contradiction, assume that S is not convex (i.e. exists $a, b, c, d \in S$, such that d is the interior of triangle spanned by a, b, c):



   Then:
$$\chi(a, b, c) = \chi(a, b, d) + \chi(a, c, d) + \chi(b, c, d) + 1 \bmod 2,$$
   contradicting the definition of S. □

8. Show that for every $r \geq 2$ there exists $n = n(r)$ such that in every coloring of $1, \ldots, n$ with r colors there exists monochromatic triple of distinct numbers satisfying $x + y = z$.

   Solution. Let $k : [n] \to [r]$, where $[n] = \{1, 2, \ldots, n$ and [r] is the set of colours, be an arbitrary coloring. Define $k' :\to [r]$ $k'(a, b) = k(|a - b|)$. From Ramsey's theorem we know that there exist n such that if we colour 2 element subsets of X $(|X| \geq$