Parabolic PDEs in Musielak - Orlicz spaces: growth controlled by a function discontinuous in time

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Non-standard growth: wrong defintion

L^p spaces: functions $f : \Omega \to \mathbb{R}$ such that L^p norm is finite:

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We generalize role of the function $M(x,\xi) = |\xi|^{p(x)}$.

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$L^M(\Omega)$ spaces: functions $f : \Omega \to \mathbb{R}^d$ such that

$$\inf\left\{\lambda > 0 : \int_\Omega M(x,u(x))\, dx \le 1\right\} < \infty.$$

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In particular, if $u \in L^M(\Omega)$ and $v \in L^{M^*}(\Omega)$ we have $u \cdot v \in L^1(\Omega)$.

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Weak solutions should have at least regularity

$$u \in L^1(0,T;W_0^{1,1}(\Omega)) \text{ and } \nabla u \in L_M((0,T)\times\Omega)$$

and satisfy weak formulation.

$p(t,x)$-Laplacian with $1 << p(t,x) << \infty$:

$$u_t = \operatorname{div}\Big[\, |\nabla u|^{p(t,x)-2}\, \nabla u \Big] + f$$

with N-function: $M(t,x,\xi) = |\xi|^{p(t,x)}$,

Examples

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double phase problems with $1 << p(t,x), q(t,x) << \infty$:

$$u_t = \operatorname{div}\Big[\, |\nabla u|^{p(t,x)-2}\, \nabla u + a(t,x)\, |\nabla u|^{q(t,x)-2}\, \nabla u \Big] + f$$

where $a(t,x)$ is a nonnegative function

Literature

- A. Elmahi, D. Meskine, *Parabolic equations in Orlicz spaces* J. London Math. Soc. 72 (2) (2005), 410–428.
- Y. Ahmida, I. Chlebicka, P. Gwiazda, A. Youssfi, *Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces*, J. Functional Analysis 275 (9) (2018), 2538-2571.
- I. Chlebicka, P. Gwiazda, A. Zatorska-Goldstein, *Parabolic equation in time and space dependent anisotropic Musielak-Orlicz spaces in absence of Lavrentiev's phenomenon*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire 36 (5) (2019), 1431-1465.
- I. Chlebicka, P. Gwiazda, A. Zatorska-Goldstein, *Renormalized solutions to parabolic equation in time and space dependent anisotropic Musielak-Orlicz spaces in absence of Lavrentiev's phenomenon*, J. Differ. Equations 267 (2) (2019), 1129-1166.

Motivating example

Consider

$$u_t = \begin{cases} \mathrm{div} \nabla u & \text{in } (0,1] \times \Omega, \\ \mathrm{div}\left(|\nabla u|^2 \nabla u\right) & \text{in } (1,2] \times \Omega, \end{cases}$$

and $u(0,x) = u_0(x)$ is given.

Typical well-posedness theory

One constructs Galerkin approximations, establishes uniform bounds and extracts weakly converging subsequences.

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Identification of the limits and uniqueness of solutions is based on the energy equalities of the form

$$\int_\Omega \left[u^2(t,x) - u_0^2(x)\right]\, dx = \int_0^t \int_\Omega f(s,x)\, u(s,x)\, dx\, ds \\ - \int_0^t \int_\Omega A(s,x,\nabla u(s,x)) \cdot \nabla u(s,x)\, dx\, ds.$$

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To obtain energy equality, one tests PDE

$$u_t(t,x) = \mathrm{div} A(t,x,\nabla u(t,x)) + f(t,x)$$

with the solution itself

Can we test with the solution?

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The troublemaker:

$$\int_{\Omega_T} \underbrace{A(t,x,\nabla u)}_{\in L_{M*}} \cdot \underbrace{[\nabla u(t,x)]^{\varepsilon_x,\varepsilon_t}}_{\in L_M???} \, dt \, dx$$

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By Young's inequality and DCT we need something like

$$\int_{\Omega_T} M\left(t, x, \frac{\nabla u(t,x) - [\nabla u(t,x)]^{\varepsilon,\delta}}{\lambda}\right) dt \, dx \to 0$$

for some $\lambda > 0$ (modular convergence).

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$$\int_0^T \int_\Omega \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, M\Big(t, \mathbf{x}, \nabla u(t, \mathbf{x} - \mathbf{y})\Big)\Big)\, dy\, dx\, dt$$

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Continuity of M in x is needed to get

$$M\left(t, \mathbf{x}, \nabla u(t, x-y)\Big)\right) \approx M\left(t, \mathbf{x} - \mathbf{y}, \nabla u(t, \mathbf{x} - \mathbf{y})\Big)\right).$$

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Moreover, continuity of M in t is needed for chain rule.

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Clearly, u^{ε_x} is smooth in spatial variable x

But also u^{ε_x} has Sobolev regularity in time $\implies$ chain rule.

Examples

Our results covers exponents roughly changing in time for:

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PARABOLIC EQUATIONS IN MUSIELAK - ORLICZ SPACES WITH DISCONTINUOUS IN TIME N-FUNCTION

MIROSLAV BULÍČEK, PIOTR GWIAZDA, AND JAKUB SKRZECZKOWSKI

ABSTRACT. We consider a parabolic PDE with Dirichlet boundary condition and monotone operator A with non-standard growth controlled by an N-function depending on time and spatial variable. We do not assume continuity in time for the N-function. Using an additional regularization effect coming from the equation, we establish the existence of weak solutions and in the particular case of isotropic N-function, we also prove their uniqueness. This general result applies to equations studied in the literature like $p(t,x)$-Laplacian and double-phase problems.

1. INTRODUCTION

1.1. **PDEs in Musielak - Orlicz spaces.** This paper focuses on study of parabolic equations having the form

$$u_t(t,x) = \mathrm{div} A(t,x,\nabla u(t,x)) + f(t,x) \text{ in } (0,T)\times\Omega, \tag{1.1}$$

completed by the homogeneous Dirichlet boundary condition and the initial value $u_0(x)$. Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain, T denotes the length of time interval, $f : (0,T)\times\Omega \to \mathbb{R}$ is a measurable bounded function and A is a monotone operator with coercivity and growth controlled by a so - called N-function $M : (0,T)\times\Omega\times\mathbb{R}^d \to \mathbb{R}$ (see Definition 1.2), i.e. for almost all $(t,x)\in(0,T)\times\Omega$ and all $\xi\in\mathbb{R}^d$, we have:

$$M(t,x,\xi) + M^*(t,x,A(t,x,\xi)) \le c\, A(t,x,\xi)\cdot\xi + h(t,x) \tag{1.2}$$

where M^* denotes the convex conjugate to M (see Definition 1.3) and $h \in L^1((0,T)\times\Omega)$. Originally, problem (1.1) was solved with $M(t,x,\xi) = |\xi|^p$ where $1 < p < \infty$. In this classical setting, (1.2) implies that A, understood as a map

$$L^p(0,T;W_0^{1,p}(\Omega)) \ni u \mapsto A(t,x,\nabla u) \in \left(L^p\left(0,T;W_0^{1,p}(\Omega)\right)\right)^*,$$

is a bounded continuous operator and standard approaches (Galerkin method and compactness in Sobolev-Bochner spaces) applies (see [5, 25] and references therein) showing that the Sobolev space is an appropriate functional setting for problem (1.1). However, if the N-function M appearing in (1.2) has not a polynomial growth with respect to ξ and is (t,x)-dependent, one has to look for a solution u such that its gradient ∇u belongs to the Musielak - Orlicz space $L_M((0,T)\times\Omega)$, i.e. the space of measureable functions $\xi : (0,T)\times\Omega \to \mathbb{R}^d$ which satisfy

$$\int_{(0,T)\times\Omega} M\left(t,x,\frac{\xi(t,x)}{\lambda}\right) \mathrm{d}t\,\mathrm{d}x < \infty$$

Key words and phrases. parabolic equation, non-standard growth, discontinuous N-function, existence, uniqueness.
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Paper “Parabolic equations in Musielak – Orlicz spaces with discontinuous in time N-function” on arXiv:1911.10826.

Other topics (not discussed today):

- existence theory ...
- Minty-Browder monotonicity trick ...
- local energy equalities ...

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- Thank you for your attention!