Measure solutions to perturbed structured population models - differentiability with respect to perturbation parameter

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Lorentz Center, Leiden, 4.12.2018

Structured population models

$$
\begin{cases}
\partial_t \mu_t + \partial_x(b(x,\mu_t)\mu_t) & = c(x,\mu_t)\mu_t \quad & \mathbb{R}^+ \times [0,T], \\
b(0,\mu_t) D_\lambda \mu_t(0) & = \int_{\mathbb{R}^+} a(x,\mu_t) d\mu_t(x) \quad & [0,T], \\
\mu_0 & = \nu & \mathbb{R}^+.
\end{cases}
$$

- a - offspring productivity
- b - how fast individuals changes their state
- c - survival chances, death rate
- $D_\lambda \mu_t(0)$ - Radon-Nikodym derivative of μ_t wrt Lebesgue measure at 0

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  - Generalization: some distributions do not have density
  - Asymptotics: some stationary distributions may not have density
  - Numerics: analysis of approximations by Dirac masses

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$$W_1(\mu,\nu) = \sup_{f \text{ is Lipschitz, } \|Df\|_\infty \leq 1} \int_{\mathbb{R}^+} f d(\mu - \nu),$$

however it is useless for not conservative problems:

$$\partial_t \mu_t + \partial_x (b(x,\mu_t)\mu_t) = c(x,\mu_t)\mu_t$$

Measure solutions: narrowly continuous, distributional solution in space of bounded nonnegative Radon measures equipped with flat metric:

$$p_F(\mu,\nu) = \sup_{f \in W^{1,\infty}, \|f\|_{W^{1,\infty}} \leq 1} \int_{\mathbb{R}^+} f d(\mu - \nu),$$

In this setting the problem is well - posed (existence, uniqueness, stability, ...)[1].

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[1] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. J. Differential Equations, 248(11):2703 – 2735, 2010.

Differentiability problem

- Recall equation:

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- Take as model functions a, b and c perturbed versions of the form

$$f^h(x, \mu) = f^0(x, \mu) + h f_p(x, \mu)$$
$$= F^0\left(x, \int_0^\infty K_{F^0}(x, y) d\mu(y)\right) + h F_P\left(x, \int_0^\infty K_{F_P}(x, y) d\mu(y)\right).$$

while initial condition is still the same. Denote solution to this problem with μ_t^h.

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while initial condition is still the same. Denote solution to this problem with μ_t^h.
- Is map $h \mapsto \mu_t^h$ differentiable and in what sense?
- Motivated, for instance, by study of optimal control of phenomena described by SPM

Counterexample

Such results **cannot** be expected in flat metric setting.

[2] P. Gwiazda, S. C. Hille, K. Łyczek, and A. Swierczewska-Gwiazda. Differentiability in perturbation parameter of measure solutions to perturbed transport equation, 2018.

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Such results **cannot** be expected in flat metric setting.
Indeed, Kamila discussed yesterday 1D transport equation:

$$\partial_t \mu_t^h + \partial_x((1+h)\mu_t^h) = 0 \qquad \mu_0^h = \delta_0. \tag{1}$$

Here, sequence $\frac{\mu_t^h - \mu_t^0}{h}$ is not convergent with respect to flat metric.

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$$\|\mu\|_Z = \sup_{\|\xi\|_{C^{1+\alpha}} \le 1} \int_{\mathbb{R}^+} \xi \, d\mu$$

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Linear case

For linear case:

$$\begin{cases} \partial_t \mu_t + \partial_x (b(x)\mu_t) & = c(x)\mu_t & \mathbb{R}^+ \times [0,T], \\ b(0) D_\lambda \mu_t(0) & = \int_{\mathbb{R}^+} a(x) d\mu_t(x) & [0,T]. \end{cases}$$

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we have formula defining solution **(SEMIGROUP PROPERTY)**:

$$\boxed{\int_{\mathbb{R}^+} \xi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi_{\xi,t}(x, 0) d\mu_0(x) \quad \text{for all } \xi \in W^{1,\infty} \cap C^1(\mathbb{R}^+),}$$

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where function $\varphi_{\xi,t}(x,s)$ satisfies **(IMPLICIT EQUATION)**:

$$\boxed{\begin{aligned} \varphi_{\xi,\mathbf{t}}(x,s) =& \xi(X_b(t-s,x)) e^{\int_0^{t-s} c(X_b(u,x)) du} \\ &+ \int_0^{t-s} a(X_b(u,x)) \varphi_{\xi,\mathbf{t}}(0, u+s) e^{\int_0^u c(X_b(v,x)) dv} du \end{aligned}}$$

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and $X_b(s,x)$ solves ODE $\frac{d}{ds}X_b(s,x) = b(X_b(s,x))$ with initial condition $X_b(0) = x$.

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$$\left\| \frac{\mu_t^{h+\Delta h_1} - \mu_t^h}{\Delta h_1} - \frac{\mu_t^{h+\Delta h_2} - \mu_t^h}{\Delta h_2} \right\|_Z =$$

$$\sup_{\|\xi\|_{C^{1+\alpha}} \le 1} \int_{\mathbb{R}^+} \xi \left(\frac{d\mu_t^{h+\Delta h_1} - d\mu_t^h}{\Delta h_1} - \frac{d\mu_t^{h+\Delta h_2} - d\mu_t^h}{\Delta h_2} \right).$$

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$$= \sup_{\|\xi\|_{C^{1+\alpha}}\leq 1}\int_{\mathbb{R}^+}\Bigg(\frac{\varphi_{\xi,t}^{h+\Delta h_1}(x,0)-\varphi_{\xi,t}^h(x,0)}{\Delta h_1} - \frac{\varphi_{\xi,t}^{h+\Delta h_2}(x,0)-\varphi_{\xi,t}^h(x,0)}{\Delta h_2}\Bigg)d\mu_0.$$

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We want to make this quantity small:

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For any f with Hölder continuous derivative on the domain of definition, one has:

$$f(y) = f(x) + f'(x)(x-y) + \underbrace{\int_0^1 \frac{d}{dt} f(ty + (1-t)x) dt - f'(x)(x-y)}_{\leq C|x-y|^{1+\alpha}}.$$

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Conclusion: we need $h \mapsto \varphi_{\xi,t}^{h}$ to have Hölder continuous derivative.

Proof of differentiability in linear case

- Recall $\varphi^h_{\xi,t}$ solves:

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\begin{aligned}
\varphi^h_{\xi,t}(x,s) &= \xi(X_{b^h}(t-s,x))e^{\int_0^{t-s} c^h(X_{b^h}(u,x))du} \\
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  where f^h denotes perturbed model function.
- Use Implicit Function Theorem in Banach spaces to have differentiability of $h \mapsto \varphi^h_{\xi,t}$.
- Differentiate implicit formula to obtain Hölder continuity of derivative.

We have proven:

Theorem

We assume:

(A1) $a^0, a_p, b^0, b_p, c^0, c_p \in C^{1,\alpha}(\mathbb{R}^+)$,

(A2) $a^h = a^0 + a_p h \geq 0$ *for any* $h \in [-\frac{1}{2}, \frac{1}{2}]$,

(A3) $b^h(0) = b^0(0) + b_p(0)h > 0$ *for any* $h \in [-\frac{1}{2}, \frac{1}{2}]$.

Consider measure solutions μ_t^h *to SPM with* $a(x) := a^h(x) = a^0(x) + ha_p(x)$, $b(x) := b^h(x) = b^0(x) + hb_p(x)$, $c(x) := c^h(x) = c^0(x) + hc_p(x)$ *and* $h \in [-\frac{1}{2}, \frac{1}{2}]$. *Then, mapping* $h \mapsto \mu_t^h$ *is Fréchet differentiable in* $C([0, T], Z)$ *where* $Z = \overline{\mathcal{M}(\mathbb{R}^+)}^{(C^{1+\alpha})^*}$. *Moreover, Fréchet derivative* $H \mapsto \frac{\partial}{\partial h}\mu_t^h|_{h=H}$ *is Hölder continuous with exponent* α.

Approximation of nonlinear equation

- Fix $k \in \mathbb{N}$.

[3] P. Gwiazda and A. Marciniak-Czochra. Structured population equations in metric spaces. *Journal of Hyperbolic Differential Equations*, 7(4):733–773, 2010.

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- Fix $k \in \mathbb{N}$.
- Divide interval $[0, T]$ for 2^k subintervals.
- For $t \in [m\frac{T}{2^k}, (m+1)\frac{T}{2^k}]$, approximation $\mu_t^{h,k}$ is defined inductively as solution to linear equation:

$$\begin{cases} \partial_t \mu_t + \partial_x (b(x, \mu_{m\frac{T}{2^k}})\mu_t) & = c(x, \mu_{m\frac{T}{2^k}})\mu_t, \\ b(0, \mu_{m\frac{T}{2^k}}) D_\lambda \mu_t(0) & = \int_{\mathbb{R}^+} a(x, \mu_{m\frac{T}{2^k}}) d\mu_t(x), \\ \mu_{m\frac{T}{2^k}} = \mu_{m\frac{T}{2^k}} \end{cases}$$

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- It was shown that $p_F(\mu_t, \mu_t^k) \to 0$ as $k \to \infty$ [3]. Hence $\|\mu_t - \mu_t^k\|_Z \to 0$ as $k \to \infty$.

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General strategy

We are interested in

$$\lim_{\Delta h \to 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h} = \lim_{\Delta h \to 0} \lim_{k \to \infty} \frac{\mu_t^{h+\Delta h,k} - \mu_t^{h,k}}{\Delta h}$$

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Theorem

Let $f_k \to f$ uniformly on a set E in some metric space (X, d). Let x be a limit point of E and suppose that $\lim_{t\to x} f_k(t) = A_k$. Then, A_k converges and $\lim_{t\to x} f(t) = \lim_{k\to\infty} A_k$. In particular,

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$$\lim_{t \to x} \lim_{n \to \infty} f_k(t) = \lim_{n \to \infty} \lim_{t \to x} f_k(t).$$

Take $E = [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$ and prove that sequence $\frac{\mu_t^{h+\Delta h,k} - \mu_t^{h,k}}{\Delta h}$:

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Differentiability of approximating sequence - k fixed

- In the first interval $[0, \frac{T}{2^k}]$ equation looks like:

$$\begin{cases} \partial_t \mu_t + \partial_x (b^h(x, \mu_0)\mu_t) & = c^h(x, \mu_0)\mu_t, \\ b^h(0, \mu_0) D_\lambda \mu_t(0) & = \int_{\mathbb{R}^+} a^h(x, \mu_0) d\mu_t(x), \\ \mu_0 = \mu_0 \end{cases}$$

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- PROBLEM: Perturbation appears in three different places: initial condition and
$a^h(x, \mu^{h,k}_{m\frac{T}{2^k}}) = a^0(x, \mu^{h,k}_{m\frac{T}{2^k}}) + h a_p(x, \mu^{h,k}_{m\frac{T}{2^k}})$

How to handle additional perturbation in nonlinearity?

Upgrade linear Theorem:

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Assume

- *(B1) Assumptions (A2) – (A3) holds for functions a, b*
- *(B2) Functions $a(h,x)$, $b(h,x)$ and $c(h,x)$ are $C^{1,\alpha}([-\frac{1}{2},\frac{1}{2}] \times \mathbb{R}^+)$ in both variables (with uniform constants in second variables).*

Consider measure solution μ_t^h of SPM with $a(x) := a(h,x)$, $b(x) = b(h,x)$, $c(x) = c(h,x)$ and $h \in [-\frac{1}{2},\frac{1}{2}]$. Then, mapping $h \mapsto \mu_t^h$ is Fréchet differentiable in $C([0,T],Z)$ where $Z = \overline{\mathcal{M}(\mathbb{R}^+)}^{(C^{1+\alpha})^*}$. Moreover, Fréchet derivative $H \mapsto \frac{\partial}{\partial h}\mu_t^h|_{h=H}$ is Hölder continuous.

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so we can use upgraded linear Theorem.

Differentiability of approximations - final proof

We split desired term:

$$\frac{\mu_t^{h+\Delta h,k} - \mu_t^{h,k}}{\Delta h} = \underbrace{\frac{\mu_t^{h+\Delta h,k} - \bar{\mu}_t^{h+\Delta h,k}}{\Delta h}}_{:=A} + \underbrace{\frac{\bar{\mu}_t^{h+\Delta h,k} - \mu_t^{h,k}}{\Delta h}}_{:=B}$$

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For A use semigroup property and induction hypothesis.

General strategy RECALL

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- **DONE!!!** converges as $\Delta h \to 0$ (differentiability of approximating sequence).
- converges uniformly for all $\Delta h \in E$ as $k \to \infty$,

Proof of uniform convergence

It is sufficient to obtain estimate:

$$\Delta^{k,t} := \sup_{\Delta h \in (-\frac{1}{2}, \frac{1}{2})} \left\| \frac{\mu_t^{h+\Delta h, k+1} - \mu_t^{h,k+1}}{\Delta h} - \frac{\mu_t^{h+\Delta h, k} - \mu_t^{h,k}}{\Delta h} \right\|_Z \leq C 2^{-k\beta}.$$

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Triangle inequalities **cannot** be used directly (we have to capture two effects simultaneously: $k \to \infty$ and $\Delta h \to 0$. We can start by writing definition

$$\Delta^{k,t} = \int_{\mathbb{R}^+} \xi \left(\frac{d\mu_t^{h+\Delta h,k+1} - d\mu_t^{h,k+1}}{\Delta h} - \frac{d\mu_t^{h+\Delta h,k} - d\mu_t^{h,k}}{\Delta h} \right)$$

RECALL: Linear case

For linear case:

$$\begin{cases} \partial_t \mu_t + \partial_x (b(x)\mu_t) & = c(x)\mu_t \qquad \mathbb{R}^+ \times [0,T], \\ b(0) D_\lambda \mu_t(0) & = \int_{\mathbb{R}^+} a(x) d\mu_t(x) \quad [0,T]. \end{cases}$$

we have formula defining solution **(SEMIGROUP PROPERTY)**:

$$\boxed{\int_{\mathbb{R}^+} \xi(x) d\mu_t(x) = \int_{\mathbb{R}^+} \varphi_{\xi,t}(x,0) d\mu_0(x) \quad \text{for all } \xi \in W^{1,\infty} \cap C^1(\mathbb{R}^+),}$$

where function $\varphi_{\xi,t}(x,s)$ satisfies **(IMPLICIT EQUATION)**:

$$\boxed{\begin{aligned} \varphi_{\xi,\mathbf{t}}(x,s) = & \xi(X_b(t-s,x)) e^{\int_0^{t-s} c(X_b(u,x)) du} \\ & + \int_0^{t-s} a(X_b(u,x)) \varphi_{\xi,\mathbf{t}}(0,u+s) e^{\int_0^u c(X_b(v,x)) dv} du \end{aligned}}$$

and $X_b(s,x)$ solves ODE $\frac{d}{ds} X_b(s,x) = b(X_b(s,x))$ with initial condition $X_b(0) = x$.

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We can apply semigroup property (I skip taking supremum over $\Delta h \in (-\frac{1}{2}, \frac{1}{2})$ and $\xi \in C^{1+\alpha}$ such that $\|\xi\| \leq 1$):

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$$= \int_{\mathbb{R}^+} \frac{\varphi_\xi^{h+\Delta h,k+1}(x,0) d\mu_{t_*}^{h+\Delta h,k+1}(x) - \varphi_\xi^{h,k+1}(x,0) d\mu_{t_*}^{h,k+1}(x)}{\Delta h}$$

$$- \frac{\varphi_\xi^{h+\Delta h,k}(x, \frac{T}{2^{k+1}}) d\mu_{t_*}^{h+\Delta h,k}(x) - \varphi_\xi^{h,k}(x, \frac{T}{2^{k+1}}) d\mu_{t_*}^{h,k}(x)}{\Delta h}$$

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Crucial observation:

$$\mu_0^{h,k} = \mu_0^{h+\Delta h,k} = \mu_0^{h,k+1} = \mu_0^{h+\Delta h,k+1} = \mu_0$$

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We can apply semigroup property until we arrive at time $t = 0$.

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$$\implies \Delta^{k,t} \leq C 2^{(1-2\alpha)k}$$

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We can apply semigroup property until we arrive at time $t = 0$.

$$\int_{\mathbb{R}^+} \xi(x) d\mu^{h,k}_{m\frac{T}{2^k}}(x) = \int_{\mathbb{R}^+} \varphi^h_{\xi,t}(x,0) d\mu^{h,k}_{(m-1)\frac{T}{2^k}}(x) =$$

$$= \int_{\mathbb{R}^+} \varphi^h_{\varphi^h_{\xi,t},t}(x,0) d\mu^{h,k}_{(m-2)\frac{T}{2^k}}(x) = \ldots = \int_{\mathbb{R}^+} \bar{\varphi}^h_{\xi,t} d\mu_0(x)$$

$$\Delta^{k,t} = \int_{\mathbb{R}^+} \frac{\bar{\varphi}^{h+\Delta h,k+1}_{\xi,t}(x,0) - \bar{\varphi}^{h,k+1}_{\xi,t}(x,0)}{\Delta h}$$

$$- \frac{\bar{\varphi}^{h+\Delta h,k}_{\xi,t}(x,0) - \bar{\varphi}^{h,k}_{\xi,t}(x,0)}{\Delta h} d\mu_0(x) =$$

$$= \int_{\mathbb{R}^+} \int_0^1 \left(\frac{\partial}{\partial H} \bar{\varphi}^{H,k+1}_{\xi,t} \Big|_{H=h+u\Delta h} - \frac{\partial}{\partial H} \varphi^{H,k}_{\xi,t} \Big|_{H=h+u\Delta h} \right) du \, d\mu_0(x)$$

$$\implies \Delta^{k,t} \leq C2^{(1-2\alpha)k} \implies \text{differentiability result for } \alpha > \tfrac{1}{2}$$

MEASURE SOLUTIONS TO PERTURBED STRUCTURED POPULATION MODELS – DIFFERENTIABILITY WITH RESPECT TO PERTURBATION PARAMETER

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Abstract. This paper is devoted to study measure solutions μ_t^h to perturbed nonlinear structured population models where t denotes time and h controlls the size of perturbation. We address differentiability of map $h \mapsto \mu_t^h$. After showing that this type of results cannot be expected in space of bounded and nonnegative Radon measures $\mathcal{M}(\mathbb{R}^+)$ equipped with flat metric, we move to slightly bigger spaces $Z = \overline{\mathcal{M}(\mathbb{R}^+)}^{(C^{1+\alpha})^*}$. We prove that when $\alpha > \frac{1}{2}$, map $h \mapsto \mu_t^h$ is differentiable in Z. The proof exploits approximation of nonlinear problem from [6] and is based on iteration of implicit integral equations obtained from study of linear equation. The result is motivated by optimal control of phenomena governed by such type of models.

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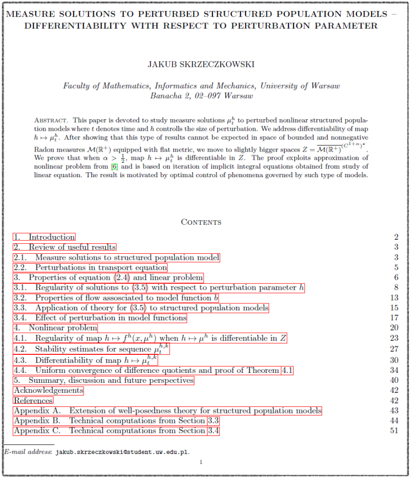

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Paper “Measure solutions to perturbed structured population models – differentiability with respect to perturbation parameter” soon on arXiv.

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- technical details, computations, ...
- iteration inequalities for solutions to implicit equations

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- Thank you for your attention!