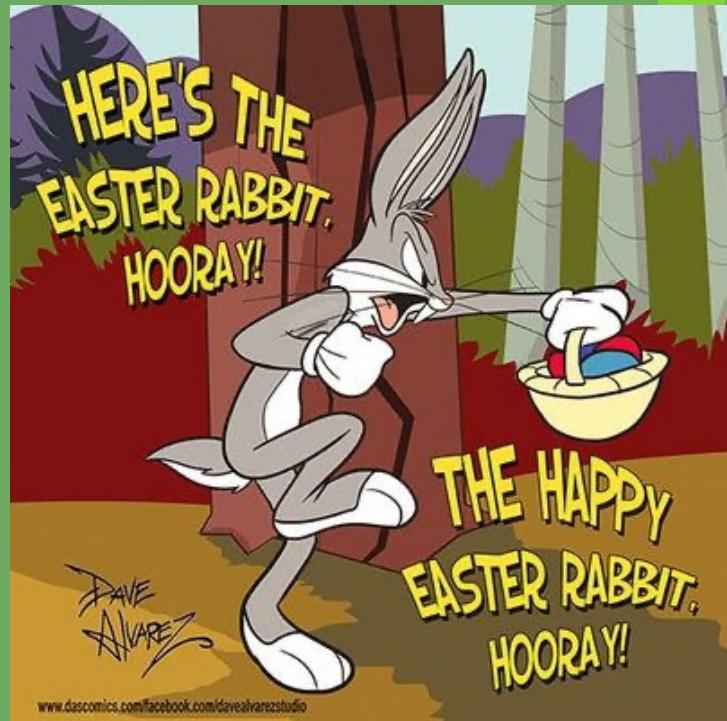


PDEs I: Tutorial 5

8.04.2021



→ gwiazdki

(u jest ciągim row. $\Delta u = 0$

$\Rightarrow u$ jest C^2 : $\Delta u = 0$ punktowo

LEMAT WEYL'A)

→ WYDARZENIA

→ Sobota 12:00

$\Phi(x)$ - rozwiązań fundametalne

" $\Delta u = 0$ ".

$$\Delta u = f$$

$$= u_f(x)$$

postulat: $u(x) = \int_{\mathbb{R}} \Phi(x-y) f(y) dy$

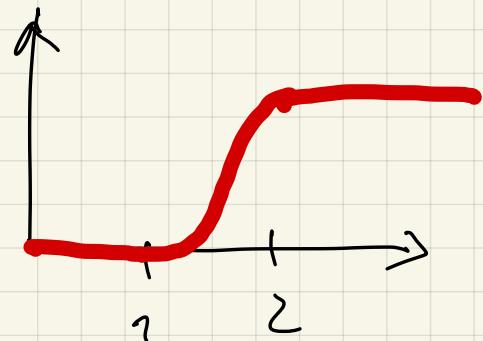
Chcemy pochodzić u

problem: $\left[\begin{array}{l} \text{nie jest rozwijalne w } \\ 0 \end{array} \right]$



Problem E1: ist die $\xi: \mathbb{R} \rightarrow \mathbb{R}$ fälsch, zeige

- $\xi(x) = 0 \quad \forall |x| \leq 1$
- $\xi(x) = 1 \quad \forall |x| \geq 2$
- $|\xi'(x)| \leq C \quad \forall x$



(LyTo)

Problem E2:

- $\zeta(s) = 0 \quad \forall |x| \leq 1$
- $\zeta(x) = 1 \quad \forall |x| \geq 2$
- $|\zeta'(x)| \leq C \quad \forall x$

↗ **funkje
ne \mathbb{R}^n**

$$\zeta_\varepsilon(x) = \begin{cases} \left(\frac{|x|^2}{\varepsilon^2}\right) & \text{↗ } \begin{array}{l} \text{betr } 1D \text{ funkje} \\ \text{betr } 1D \text{ funkje} \end{array} \end{cases}$$

$$\bullet \zeta_\varepsilon(0) = 0 \quad \frac{|x|^2}{\varepsilon^2} \leq 1 \iff |x| \leq \varepsilon$$

$$\bullet \zeta_\varepsilon(x) = 1 \quad |x| \geq \sqrt{\varepsilon} \quad \rightarrow |x|^2 = x_1^2 + \dots + x_n^2$$

$$\nabla \zeta_\varepsilon(x) = \zeta' \left(\frac{|x|^2}{\varepsilon^2} \right) \cdot \frac{2x}{\varepsilon^2}$$

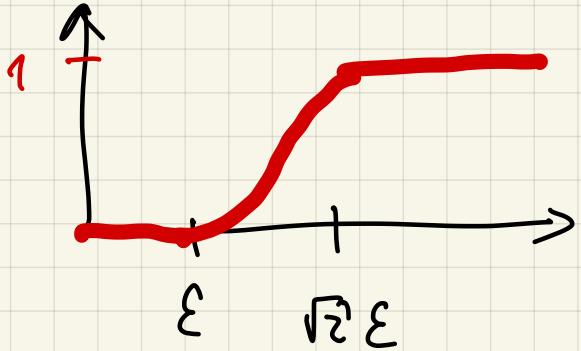
$$\nabla \zeta_\varepsilon(x) = \zeta' \left(\frac{|x|^2}{\varepsilon^2} \right) - \frac{x}{\varepsilon^2} \quad x = (x_1, \dots, x_n)$$

zu. $|\nabla \zeta_\varepsilon(x)| \leq \frac{C}{\varepsilon}$. (C wie zahl auf ε)

$$\zeta'(y) \neq 0 \quad \text{dля } |y| \leq 2$$

$$\zeta' \left(\frac{|x|^2}{\varepsilon^2} \right) \neq 0 \quad \text{для } \frac{|x|}{\varepsilon} \leq \sqrt{2}$$

$$|\nabla \zeta_\varepsilon(x)| \leq \begin{cases} C \frac{\sqrt{2} \varepsilon}{\varepsilon^2} \leq \tilde{C} \frac{1}{\varepsilon} & |x| \leq \sqrt{2} \varepsilon \\ 0 \leq \tilde{C} \frac{1}{\varepsilon} & |x| > \sqrt{2} \varepsilon \end{cases}$$



$$(\forall \varepsilon) \mid \leq \frac{C}{\varepsilon}.$$

□.

Problem E3:

$$W_f(x) = \int_{\mathbb{R}} \phi(x-y) f(y) dy$$

$$W_f^\varepsilon(x) := \int_{\mathbb{R}} \underbrace{\phi(x-y)}_{\substack{\text{nur } \varepsilon \text{-mich} \\ \text{gilt } x=y}} \underbrace{\sum_{y \in \varepsilon} \phi(x-y)}_{\mid x-y \mid \leq \varepsilon} f(y) dy$$

$$\underline{\text{Ziel}}: \quad w_f^\varepsilon \xrightarrow{\sim} u_f \quad (f \in L^\infty(\Omega))$$

$$|w_f^\varepsilon(x) - u_f(x)| = \left| \int_{\Omega} \Phi(x-y) f(y) \underbrace{\left[1 - \zeta_\varepsilon(x-y) \right]}_{=0} dy \right|$$

$$\leq \|f\|_\infty$$

$$= \left| \int_{\Omega} \Phi(x-y) f(y) \underbrace{\left(1 - \zeta_\varepsilon(x-y) \right)}_{\substack{|x-y| \leq \sqrt{2}\varepsilon \\ \leq 1}} dy \right|$$

$$\text{gdy } \zeta_\varepsilon(x-y) = 1$$

$$\text{gdy } |x-y| \geq \sqrt{2}\varepsilon$$

$$\leq \|f\|_\infty \int_{|x-y| \leq \sqrt{2}\varepsilon} |\Phi(x-y)| dy$$

$$\leq C \int_{|x-y| \leq \sqrt{2}\varepsilon} |\Phi(x-y)| dy$$

"friending
o materscii"

$n \geq 3$: $\Phi(x-y) = \frac{C}{|x-y|^{n-2}}$

$$\int_{|x-y| \leq \sqrt{2}\varepsilon} \frac{C}{|x-y|^{n-2}} dy = \int_{B(x, \sqrt{2}\varepsilon)} \frac{C}{|x-y|^{n-2}} dy$$


$$= \int_{B(0, \sqrt{2}\varepsilon)} \frac{C}{|y|^{n-2}} dy = \int_0^{\sqrt{2}\varepsilon} \int_{\partial B(0, r)} \frac{C}{r^{n-2}} g(S(y)) dr$$

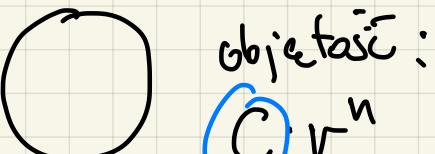
$$\int_0^{\sqrt{2}\varepsilon} \int_{\partial B(q, r)} \frac{C}{r^{n-2}} g(S(y)) dr =$$

$$= \int_0^{\sqrt{2}\varepsilon} \frac{C r^{n-1}}{r^{n-2}} dr =$$

$$= \int_0^{\sqrt{2}\varepsilon} C \cdot r dr = \frac{C}{2} (\sqrt{2}\varepsilon)^2 \rightarrow 0$$

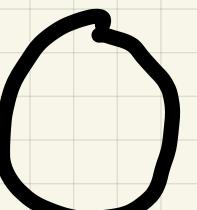
put $\varepsilon \rightarrow 0$.

Geometrie III



$$\begin{aligned} 2D: & \pi r^2 \\ 3D: & \frac{4}{3} \pi r^3 \dots \end{aligned}$$

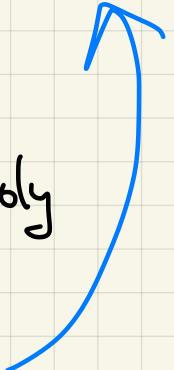
objekt kuli
rechnung



pole pow:
 $\tilde{C} \cdot r^{n-1}$

$$\begin{aligned} 2D: & 2\pi r \\ 3D: & 4\pi r^2 \end{aligned}$$

$$\underline{n=2}: \quad \Phi(x-y) \simeq C \log|x-y|$$

$$\int_{|x-y| \leq \sqrt{2}\varepsilon} \log|x-y| \, dy = \int_{B(x, \sqrt{2}\varepsilon)} \log|x-y| \, dy$$


$$= \int_{B(0, \sqrt{2}\varepsilon)} \log|y| \, dy = \int_0^{\sqrt{2}\varepsilon} \int_{\partial B(0, r)} \log r \, g(S(y)) \, dr$$

$$= \int_0^{\sqrt{2}\varepsilon} \log r \cdot r \, dr \rightarrow 0 \quad (\varepsilon \rightarrow 0) \quad (\text{Wolfram})$$

$$w_f(x) = \int_{\mathbb{R}} \phi(x-y) f(y) dy$$

$$w_f^\varepsilon \xrightarrow{\varepsilon} w_f$$

$$w_f^\varepsilon(x) := \int \underbrace{\phi(x-y)}_{\text{red bracket}} \underbrace{\zeta_\varepsilon(x-y)}_{\text{red bracket}} f(y) dy$$

Problem E4: $w_f \in C^1(\mathbb{R})$

$$f \in L^\infty$$

i $D w_f(x) = \int_{\mathbb{R}} D\phi(x-y) f(y) dy.$

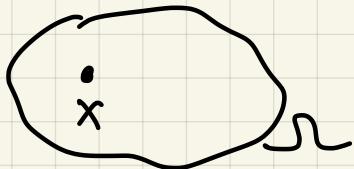
Rozw:

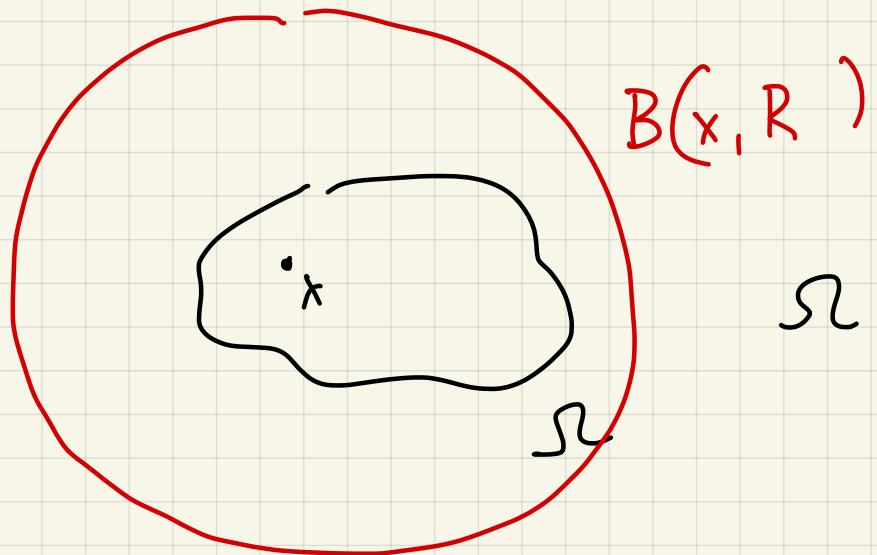
$$\underline{\Phi} \sim |x-y|^{-(n-2)}$$

Krok 1: $x \mapsto \int_{\mathbb{R}} D\underline{\Phi}(x-y) f(y) dy$ jest dobrze zdef

$$|D\underline{\Phi}(x-y)| \leq \frac{C}{|x-y|^{n-1}}$$

$$\left| \int_{\mathbb{R}} D\underline{\Phi}(x-y) f(y) dy \right| \leq \|f\|_{\infty} \int_{\mathbb{R}} \frac{C}{|x-y|^{n-1}} dy$$





$$S \subset B(x, R)$$

$$\begin{aligned} \int_S \frac{C}{|x-y|^{n-1}} dy &\leq \int_{B(x, R)} \frac{C}{|x-y|^{n-1}} dy = \\ &= \int_{B(0, R)} \frac{C}{|y|^{n-1}} dy = \int_0^R \int_{\partial B(0, r)} \frac{C}{r^{n-1}} dS(y) dr \end{aligned}$$

$$\int_0^R \int_{\partial B(0,r)} \frac{C}{r^{n-1}} dS(y) dr =$$

$$= \int_0^R \frac{\tilde{C} r^{n-1}}{r^{n-1}} dr = \tilde{C} R < \infty.$$

gładkie funkcje, ograniczone

Krok 2: $w_f^\varphi(x) = \int_{\mathbb{R}} \underbrace{\hat{\phi}(x-y)}_{\text{funkcja gładka}} \underbrace{\hat{\chi}^\varepsilon(x-y)}_{\text{funkcja gładka, ograniczona}} f(y) dy$

$$Dw_f^\varphi(x) = \int_{\mathbb{R}} D\hat{\phi}(x-y) \hat{\chi}^\varepsilon(x-y) f(y) dy$$

$$+ \int_{\mathbb{R}} \hat{\phi}(x-y) D\hat{\chi}^\varepsilon(x-y) f(y) dy$$

$$\text{Def: } D w_f^\varepsilon(x) \Rightarrow \int_{\mathbb{R}} D\phi(x-y) f(y) dy$$

$$D w_f^\varepsilon(x) = \int_{\mathbb{R}} D\phi(x-y) \{^\varepsilon(x-y) f(y) dy$$

$$+ \int_{\mathbb{R}} \phi(x-y) D\{^\varepsilon(x-y) f(y) dy$$

$$\left| \int_{\mathbb{R}} \phi(x-y) D\{^\varepsilon(x-y) f(y) dy \right| \leq$$

$$\neq 0 \text{ s.t. } \varepsilon \leq |x-y| \leq \sqrt{\varepsilon} \varepsilon$$

$$\int_{\{|x-y| \leq \sqrt{\varepsilon}\}} |\phi(x-y)| dy \cdot \frac{C \|f\|_\infty}{\varepsilon}$$



$$\sum_{\substack{S \subseteq \{x-y\} \\ |\sum S|}} \lesssim \sum_{\substack{S \\ |\sum S|}} |S(x-y)| \log^+ |S(x-y)| \cdot \frac{C \|f\|_\infty}{\varepsilon} \quad (n \geq 3)$$

$$= \int_{-\varepsilon}^{\sqrt{2}\varepsilon} \int_{\partial B(0,r)} \frac{C}{r^{n-2}} \circ (S(y)) dr \frac{C \|f\|_\infty}{\varepsilon}$$

- presunut term do 0
- prostekokonanie
- $|y|=r$ na $\partial B(0,r)$

$$= \int_{-\varepsilon}^{\sqrt{2}\varepsilon} \frac{C}{r^{n-2}} r^{n-1} dr \frac{C \|f\|_\infty}{\varepsilon}$$

$$= \tilde{C} \frac{\|f\|_\infty}{\varepsilon} \varepsilon^2 = \tilde{C} \varepsilon \rightarrow 0 \text{ przy } \varepsilon \rightarrow 0.$$

Ostatni cel:

$$\int_{\mathbb{R}} D\phi(x-y) f(y) dy$$

$$\int_{\mathbb{R}} D\phi(x-y) \zeta^\varepsilon(x-y) f(y) dy$$

≤ 1

$$\left| \int_{\mathbb{R}} D\phi(x-y) f(y) \left[1 - \zeta^\varepsilon(x-y) \right] dy \right| = 0 \text{ gdy } |x-y| \geq \sqrt{2} \varepsilon$$

$$\leq \|f\|_\infty \int_{|x-y| \leq \sqrt{2}\varepsilon} D\phi(x-y) dy \frac{C}{|x-y|^{n-1}}$$

$$\|f\|_{\infty} \int_{\{|x-y| \leq \sqrt{r}\}} D\Phi(x-y) dy$$

$$\leq C \|f\|_{\infty} \int_0^{\sqrt{r}\varepsilon} \int_{\partial B(0,r)} \frac{C}{r^{n-1}} dS(y) dr =$$

$$= \tilde{C} \|f\|_{\infty} \int_0^{\sqrt{r}\varepsilon} dr = \tilde{C} \|f\|_{\infty} \sqrt{r}\varepsilon \rightarrow 0.$$

$$w_f^\varepsilon(x) \Rightarrow \int_{\mathbb{R}} \hat{\phi}(x-y) f(y) dy.$$

$$\therefore D w_f^\varepsilon(x) \Rightarrow \int_{\mathbb{R}} D\hat{\phi}(x-y) f(y) dy$$

$$w_f^\varepsilon \in C^1(\mathbb{R})$$

w_f^ε jest ciągiem Cauchy'ego w $C^1(\mathbb{R})$

$\Rightarrow w_f^\varepsilon$ ma granicę w $C^1(\mathbb{R})$

$$\Rightarrow \int_{\mathbb{R}} \hat{\phi}(x-y) f(y) dy \in C^1(\mathbb{R})$$

Problem E5 $\Rightarrow \uparrow$.

$$Dw_f(x) = \int_{\mathbb{R}} D\Phi(x-y) f(y) dy$$

Kontrololat $\partial_i \partial_j u_f(x) \stackrel{?}{=} \int_{\mathbb{R}} \underbrace{\partial_i \partial_j \Phi(x-y)}_{\leq \frac{C}{|x-y|^n}} f(y) dy$

$$\|f\|_{\infty} \int_0^R \frac{C}{r^n} r^{n-1} dr = \|f\|_{\infty} \int_0^R \frac{C}{r} dr \cancel{\propto} \infty$$

$f \in L^\infty$ albo f jest ciągła

nie wystarczy

(Schauder's estimate) $f \in C^\alpha$

$$\exists_C |f(x) - f(y)| \leq C |x-y|^\alpha \quad \alpha \in (0,1]$$

(↑)

$$-\Delta u = f$$

$$f \in C^\alpha \Rightarrow u \in C^{2+\alpha}$$

(echcielibysmy)

Sobota:
Daniel
+ Michał

$$f \in L^2 \Rightarrow u \in H^2$$

(
u ma dwie pochodne
tylko w "Stabym" sensie)

Równanie ciepła:

$$u_t - \Delta u = 0$$

↑
pochodna po czasie

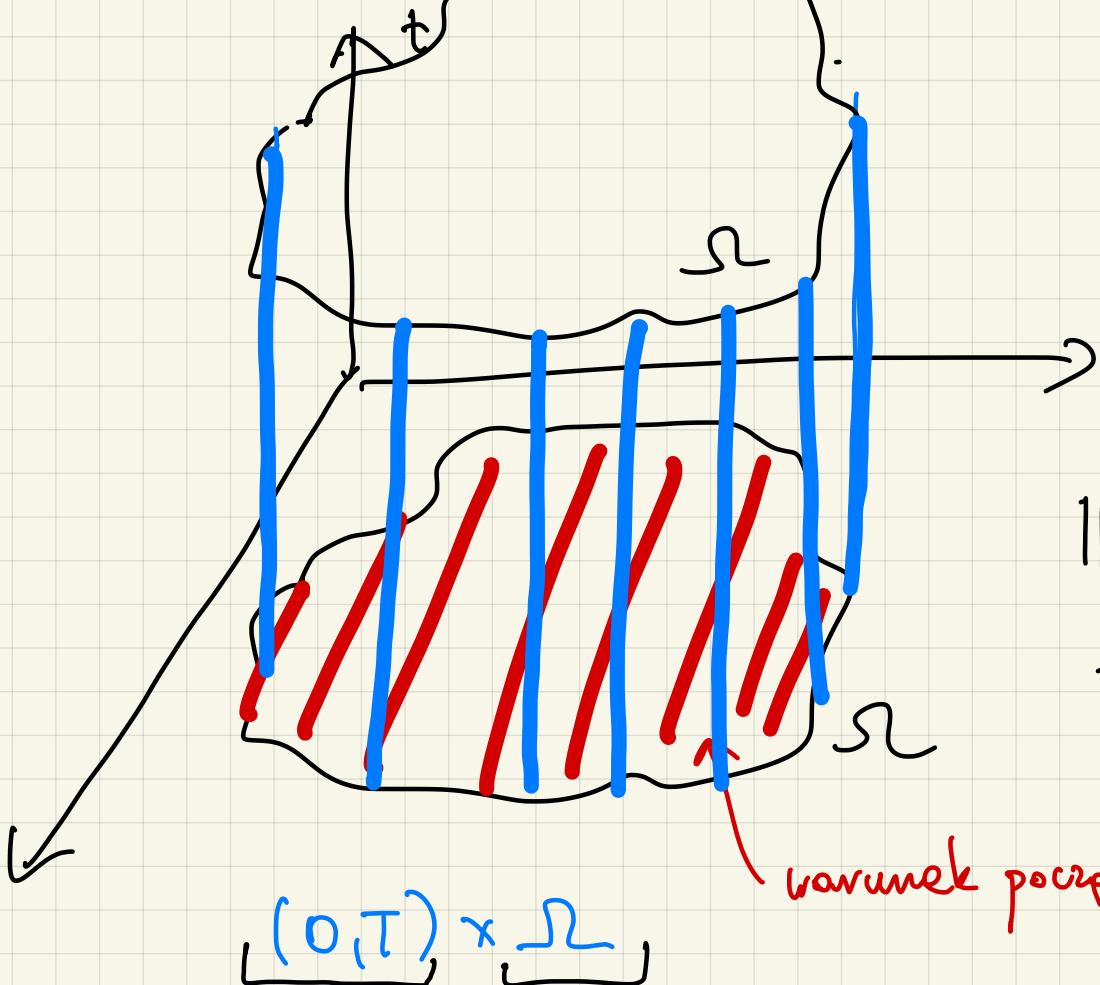
↗ Laplasjan

→ Równanie ciepła

→ Równanie reakcji-dyfuzji (reakcje chemiczne)

→ Równanie Naviera-Stokesa:

$$u_t - \Delta u = \operatorname{div}(u \otimes u) + p,$$



warunk początkowy

$$\begin{aligned} \mathbb{R}^n \\ \omega \subset \mathbb{R}^n \end{aligned}$$

$$\begin{cases} u_t - \Delta u = 0 & \text{na } (0,T) \times \Omega \\ u(0,x) = u_0(x) & \text{dla } x \in \Omega \\ u(t,x) = g(x) & \text{dla } x \in \partial\Omega \end{cases}$$

Zadanie 1 (III seria)

$$u(t, x) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x|^2}{2t}}$$

ogłosić $N(0, t)$

$$\frac{x_1^2 + \dots + x_n^2}{2t}$$

$$N(0, \sigma^2)$$

$$\sigma^2 = t$$

$$(x^k)^{\frac{1}{k}} = kx^{k-1}$$

$$(A) \quad u_t - \Delta u = 0 \quad t > 0$$

$$u_t = \frac{-\frac{n}{2} \cdot 2\pi}{(2\pi t)^{n/2+1}} e^{-\frac{|x|^2}{2t}} + \underbrace{\frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x|^2}{2t}} \cdot \frac{|x|^2}{2t^2}}_{= u}$$

$$= u \left[-\frac{n\pi}{2\pi t} + \frac{|x|^2}{2t^2} \right] = u \left[-\frac{n}{2t} + \frac{|x|^2}{2t^2} \right]$$

$$u_t = u \left[-\frac{u}{2t} + \frac{|x|^2}{2t^2} \right] \quad \Delta u = ?$$

$$\partial_{x_i} u = u \left(-\frac{2x_i}{2t} \right) = -u \frac{x_i}{t}$$

$$\partial_{x_i}^2 u = -\frac{x_i}{t} \partial_{x_i} u - u \frac{1}{t} =$$

$$= -\frac{x_i}{t} u \left(-\frac{x_i}{t} \right) - \frac{u}{t} =$$

$$= \frac{x_i^2}{t^2} u - \frac{u}{t}$$

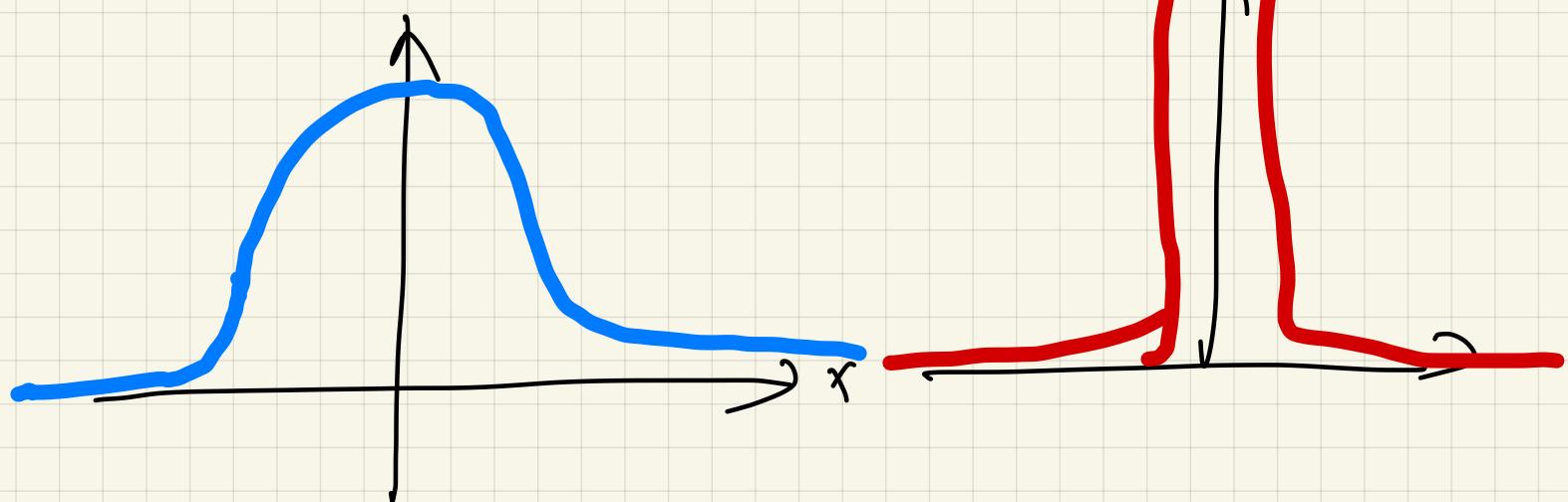
$$\Delta u = \frac{|x|^2 u}{t^2} - \frac{n u}{t}$$

go to
sprawdz

$$u(t, x) = \frac{1}{(2\pi t)^{1/2}} e^{-|x|^2/2t}$$

matte +

unmeasured +



RPTII (stosunek zmienności miar: $\{\mu_n\} \subset \mathcal{P}(\mathbb{R}^n)$)

$\mu_n \Rightarrow \mu$ gdy \forall
f ciągiej,
ograniczej

$$\int f d\mu_n \rightarrow \int f d\mu.$$

$N(0, t) \Rightarrow \delta_0$ gdy $t \rightarrow 0$.

(za tydzien)

Ogólna prawały fizyki PDFs:

wzajemne oddziaływanie merytorme
są dane pozytywne (masy, funkcje $\sim L^1$)

$\Rightarrow f$ wzajemne jest gęste
 $t > 0$

amb.: (wymianie ciepła generuje przegrzeć możliwości)