Introduction to PDEs (SS 20/21)

Homeworks

Compiled on 24/05/2021 at 9:49pm

General instruction: Problems have to be solved in groups of 2 students and the solutions have to be submitted via moodle before the class begins (10:15).

It is highly recommended to submit your solutions in English (or French).

Contents

Homework 1: problems for 11/03/2021

1. Let $b : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 and Lipschitz vector field. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a C^1 and Lipschitz function. Use Problems 4 and 5 (PS A1) to establish well-posedness theory for C^1 solutions to equation
$$\partial_t u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0, \qquad u(0,x) = u_0(x)$$
similarly as in Problem 1 (PS A1). Comment on existence, uniqueness, stability with respect to vector field b as well as initial data u_0, maximum principle and semigroup property.

2. A typical feature of nonlinear hyperbolic equations is formation of discontinuities or singularities i.e. even if one starts with a smooth initial condition, the solution becomes discontinuous in a finite time. As an example, consider Burger's equation
$$u_t + u\, u_x = 0, \qquad u(0,x) = u_0(x),$$
where $u(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$.

   (A) Show that characteristics method implies implicit equation $u(t, u_0(x)\, t + x) = u_0(x)$.

   (B) Find solution to Burger's equation with $u_0(x) = 1 - x$ for $0 \le t < 1$. What happens at $t = 1$?

Homework 2: problems for 18/03/2021

1. We establish a connection between harmonic and holomorphic functions. In what follows we identify $\mathbb{C}$ with $\mathbb{R}^2$. For $\Omega \subset \mathbb{C}$, we write $\widetilde{\Omega}$ for the subset of $\mathbb{R}^2$ corresponding to Ω.

   (A) Prove that if $u : \Omega \to \mathbb{C}$ is holomorphic then real and imaginary parts of u are harmonic functions as maps from $\widetilde{\Omega}$ to $\mathbb{R}$. *Hint:* use Cauchy-Riemann equations.

   (B) Conversely, assume that Ω is simply connected and let $u : \widetilde{\Omega} \to \mathbb{R}$ be a harmonic function. Prove that there is a holomorphic function $v : \Omega \to \mathbb{C}$ such that real part of v equals u. *Hint:* Consider complex derivative of u, namely $w(x+iy) = u_x(x,y) - i\, u_y(x,y)$ and use path integration to find its antiderivative. Observe that u is the real part of the antiderivative.

2. In what follows we establish comparison, maximum principle and stability for Poisson's equation. As always we assume $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^n$ open, connected and bounded. Moreover, we assume that u_1, u_2 solve

$$\begin{cases} -\Delta u = f_1 & \text{in } \Omega \\ u = g_1 & \text{on } \partial\Omega \end{cases}, \qquad \begin{cases} -\Delta v = f_2 & \text{in } \Omega \\ v = g_2 & \text{on } \partial\Omega \end{cases},$$

   where $f_1, f_2 \in C(\Omega)$ and $g_1, g_2 \in C(\partial\Omega)$.

   (A) (comparison principle) Suppose that $f_1 \le f_2$, $g_1 \le g_2$. Then, $u_1 \le u_2$.

   (B) (maximum principle) We have

$$\|u\|_{L^\infty(\Omega)} \le C\left(\|f_1\|_{L^\infty(\Omega)} + \|g_1\|_{L^\infty(\partial\Omega)}\right),$$

   where C is a constant that depends only on the size of Ω. *Hint:* Consider $\widetilde{u}(x) = \frac{u(x)}{\|f_1\|_{L^\infty(\Omega)} + \|g_1\|_{L^\infty(\partial\Omega)}}$ and $w(x) = \frac{M - x_1^2}{2} + 1$ for appropriate M. Apply (A) to w and $\widetilde{u}$.

   (C) (stability) Deduce from (B) that

$$\|u - v\|_{L^\infty(\Omega)} \le C\left(\|f_1 - f_2\|_{L^\infty(\Omega)} + \|g_1 - g_2\|_{L^\infty(\partial\Omega)}\right).$$

Homework 3: problems for 25/03/2021

1. In this exercise we find measure solution to the PDE (called continuity equation)

$$\partial_t \mu_t + \partial_x (b(x)\mu_t) = 0.$$

with initial condition μ_0. Recall that $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a measure solution if for all test functions $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R})$ we have

$$\int_{\mathbb{R}^+\times\mathbb{R}} \partial_t\varphi(t,x)\,\mathrm{d}\mu_t(x)\mathrm{d}t \; + \int_{\mathbb{R}^+\times\mathbb{R}} \partial_x\varphi(t,x)\,b(x)\,\mathrm{d}\mu_t(x)\mathrm{d}t + \int_{\mathbb{R}} \varphi(0,x)\,\mathrm{d}\mu_0(x) = 0$$

This will generalize the case $b(x) = b$ known from the class. The idea is to define appropriate transport operator acting on measures.

(A) Let (X_1, Σ_1), (X_2, Σ_2) be two measure spaces (here Σ_1, Σ_2 denotes σ-algebras of subsets of X_1 and X_2 respectively). Let μ be a nonnegative measure on (X_1, Σ_1) and T be a measurable map $T : (X_1, \Sigma_1) \to (X_2, \Sigma_2)$. Prove that

$$T^{\#}\mu(A) = \mu(T^{-1}(A))$$

defines a nonnegative measure on (X_2, Σ_2). We say that $T^{\#}\mu$ is a push-forward of μ along T.

(B) Let μ be a bounded (i.e. $\mu(X_1) < \infty$) and nonnegative measure. Prove change-of-variables formula: for all bounded $f : X_2 \to \mathbb{R}$ we have

$$\int_{X_2} f(x)\;\mathrm{d}T^{\#}\mu(x) = \int_{X_1} f(T(x))\;\;\mathrm{d}\mu(x).$$

Hint: first, consider simple function $f(x) = \mathbb{1}_A(x)$ where $A \in \Sigma_2$. Then, prove the result for all nonnegative bounded functions and finally for all bounded.

(C) Let μ_0 be a nonnegative bounded measure on $\mathbb{R}$ and let $X_b(t,x) : \mathbb{R} \to \mathbb{R}$ be the flow of the vector field b. Prove that $\mu_t := X_b(t,x)^{\#}\mu_0$ is a measure solution to the continuity equation.

2. In the next class we will prove that given $g \in C(\partial B_R(x))$ there exists unique

$$u \in C^2(B_R(x)) \cap C(\overline{B_R(x)})$$

such that $\Delta u = 0$ in $B_R(x)$ and $u = g$ on $\partial B_R(x)$. Use this to prove the following.

(A) Let $u \in C(\Omega)$. Prove that u is harmonic in Ω if and only if for all balls $B_R(y)$ compactly contained in Ω

$$u(y) = \fint_{\partial B_R(y)} u(x).$$

Hence, mean value property upgrades regularity of u from $C(\Omega)$ to $C^2(\Omega)$!!!

Hints: (1) Fix ball $B_R(y)$ and use solvability of Laplace equation to choose harmonic function with appropriate boundary value. (2) Observe that mean value property implies all maximum principle results.

(B) Conclude that the limit of a uniformly convergent sequence of harmonic functions is harmonic.

Homework 4: problems for 8/04/2021

Problem 1 is worth 1.5 point while Problem 2 is worth 0.5 point.

1. (mollifiers once again) Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a smooth nonnegative function supported on the unit ball $B_1(0)$ such that $\int_{\mathbb{R}^d} \eta(x)\, \mathrm{d}x = 1$. We define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\, \eta\left(\frac{x}{\varepsilon}\right).$$

For any $f : \mathbb{R}^d \to \mathbb{R}$ we write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ if $f \in L^1(K)$ for all compact sets $K \subset \mathbb{R}^d$. Similarly we define $L^p_{\text{loc}}(\mathbb{R}^d)$. Finally, if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define

$$f * \eta_\varepsilon = \int_{\mathbb{R}^d} f(y)\, \eta_\varepsilon(x-y)\, \mathrm{d}y.$$

(A) Give an example of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ but $f \notin L^1(\mathbb{R}^d)$.

(B) Prove inclusions $L^p(\mathbb{R}^d) \subset L^p_{\text{loc}}(\mathbb{R}^d) \subset L^1_{\text{loc}}(\mathbb{R}^d)$.[1]

(C) Function η_ε is again a smooth nonnegative function supported on the unit ball $B_\varepsilon(0)$ such that $\int_{\mathbb{R}^d} \eta_\varepsilon(x)\, \mathrm{d}x = 1$.

(D) If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ then $f * \eta_\varepsilon$ is smooth.

(E) If f is continuous then $f * \eta_\varepsilon \to f$ uniformly on compact subsets of $\mathbb{R}^d$.

(F) If $f \in L^p(\mathbb{R}^d)$ then $f * \eta_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$.

(G) If $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ then $f * \eta_\varepsilon \to f$ in $L^p(K)$ for all compact $K \subset \mathbb{R}^d$.

Hints and comments: (D) follows from differentiation under integral while (E) uses (C). Part (F) requires density of continuous compactly supported functions in $L^p(\mathbb{R}^d)$ and Young's convolution inequality which may be assumed without the proof. Parts (E) and (F) may be found in Brezis (Proposition 4.21, Theorem 4.22) which you should consult in case of troubles. Part (G) is an easy adaptaion of argument in (F).

2. Let $\Phi(x)$ be the scaled fundamental solution to Laplace equation

$$\Phi(x) = \begin{cases} \frac{1}{n\,(2-n)\,\alpha_n}\, |x|^{2-n} & \text{if } n > 2, \\ -\frac{1}{2\pi} \log|x| & \text{if } n = 2. \end{cases}$$

Prove estimates

$$|D_i \Phi(x)| \le \frac{1}{n\,\alpha_n} |x|^{1-n}, \qquad |D_{i,j}\Phi(x)| \le \frac{1}{\alpha_n} |x|^{-n}$$

[1] This is why most theorems are formulated for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ as this is the most general setting.

Homework 5: problems for 15/04/2021

1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $f \in L^\infty(\Omega) \cap C^\alpha(\Omega)$ for some $\alpha \in (0,1]$. Consider Ω_0 such that $\Omega \subset \Omega_0$ and extend $f = 0$ in $\Omega_0 \setminus \Omega$. Prove that $w_f \in C^2(\Omega)$ and

$$D_{i,j} w_f(x) = \int_{\Omega_0} D_{i,j}\Phi(x-y)\,(f(x) - f(y))\,\mathrm{d}y - f(x)\int_{\partial\Omega_0} D_i\Phi(x-y)\,n_j(y)\,\mathrm{d}S(y)$$

where n_j is the j-th component of $\mathbf{n}$ and w_f is the Newtonian potential of f.

Steps:

(A) If $f \in C^\alpha(\Omega)$, then there is a constant C_f such that for all $x, y \in \Omega$,

$$|f(x) - f(y)| \le C\,|x-y|^\alpha.$$

Use this to prove that the function above (candidate for the second derivative) is well-defined (finite).

(B) As when computing first derivative, use function ξ_ε to remove singularity around $x \approx y$ and consider $v_\varepsilon(x) = \int_\Omega D_i\Phi(x-y)\,\xi_\varepsilon(x-y)\,f(y)\,\mathrm{d}y$. Observe that Ω may be replaced with Ω_0 in the definition of $v_\varepsilon(x)$.

(C) Compute $D_j v_\varepsilon$ and split $f(y) = (f(y) - f(x)) + f(x)$. Use integration by parts in the second term. What can be said about ξ_ε in the second term for small ε?

(D) Prove that $D_j v_\varepsilon$ converges uniformly on compact subsets of Ω to the candidate for $D_{i,j} w_f(x)$. Combine this with uniform convergence of v_ε to $D_i w_f(x)$ and with the fact that C^2 is a Banach space to conclude the proof.

2. (energy method for uniqueness to heat equation) Let $u(t,x)$ be a classical solution to

$$\begin{aligned} u_t - \Delta u &= f(x) \text{ in } [0,T]\times\Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ u(t,x) &= g(x) \text{ for } x \in \partial\Omega \end{aligned} \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that

$$\partial_t u(t,x) \in C([0,T]\times\Omega), \qquad \partial_i\partial_j u(t,x) \in C([0,T]\times\Omega), \qquad i,j = 1,...,n,$$

i.e. u is C^1 in time and C^2 in space.

(A) Let $f(x) = g(x) = 0$. Define *energy* with $E(t) = \int_\Omega |u(t,x)|^2\,\mathrm{d}x$. Prove that $E(t)$ is nonincreasing.

(B) Deduce uniqueness for solutions to (1).

We did a similar thing for Poisson equation.

Homework 6: problems for 22/04/2021

This week, there are two small problems but see below!

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In the class we have seen the following interpolation inequality between Holder spaces: for all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that for all $u \in C^{2,\alpha}(\Omega)$
$$\|u\|_{C^2(\Omega)} \leq C(\varepsilon)\, \|u\|_\infty + \varepsilon\, [D^2 u]_{C^\alpha(\Omega)}.$$
Generalize it to the case of L^p spaces, i.e.
$$\|u\|_{C^2(\Omega)} \leq C(\varepsilon)\, \|u\|_{L^p(\Omega)} + \varepsilon\, [D^2 u]_{C^\alpha(\Omega)}.$$

2. (maximum principle for porous media equation) Let $u(t,x)$ be a smooth functon satisfying
$$u_t - \Delta F(u) \leq 0 \text{ in } [0,T] \times \Omega$$
on the bounded domain Ω. Here, F is a strictly increasing function ($F'(\lambda) > 0$ for all λ). Prove that u attains its maximum either at $t = 0$ or $x \in \partial\Omega$ (i.e. on the so-called parabolic boundary). Deduce uniqueness for porous media equation:
$$\begin{aligned} u_t - \Delta F(u) &= f(x) \text{ in } [0,T] \times \Omega, \\ u(0,x) &= u_0(x) \text{ for } x \in \Omega, \\ u(t,x) &= g(x) \text{ for } x \in \partial\Omega \end{aligned} \tag{2}$$

Additional task:
In the assigned groups, prepare 2-3 minutes talk and blackboard outlining the most important things about:

(A) transport equations, hyperbolic equations in general,

(B) Laplace and Poisson equation: everything without existence,

(C) Laplace and Poisson equation: existence using Green's function, regularity,

(D) heat equation, parabolic equations in general.

Have a look at both lecture and tutorial material to prepare for the final exam.

Homework 7: problems for 29/04/2021

This week, there are four quick problems for getting familiar with distributions and distributional derivatives.

1. Let $k \in \mathbb{N}$ and $x_0 \in \Omega \subset \mathbb{R}$. Prove that $T_k(\varphi) = \varphi^{(k)}(x_0)$ (i.e. k-th derivative at x_0) is a distribution and find its degree.

2. Prove that the formula $T(\varphi) = \sum_{k=1}^{\infty} \varphi^{(k)}(1/k)$ defines a distribution on $\Omega = (0, \infty)$. Find its degree.

3. Compute distributional derivative of the function $\mathbb{1}_{x>0}$ on $(-1, 1)$.

4. Prove that distributional derivatives satisfy Schwarz lemma (their order can be interchanged).

Homework 8: problems for 6/05/2021

1. Consider $u(x) = \mathbb{1}_{|x|<1}$ in $\mathbb{R}^d$.

   (A) Prove that $u \in W^{1,p}(B_1(0))$ where $B_1(0) = \{|x| < 1\}$ and find its weak gradient.

   (B) Prove that $u \notin W^{1,p}(\mathbb{R}^d)$.

   (C) Find distributional gradient of u in $\mathbb{R}^d$, i.e. find formula for the distribution $(\partial_{x_i} T_u)(\phi)$ where $T_u(\phi) = \int_{\mathbb{R}^d} u(x)\,\phi(x)\mathrm{d}x$. Is there a function $v_i \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$(\partial_{x_i} T_u)(\phi) = \int_{\mathbb{R}^d} v_i(x)\,\phi(x)\mathrm{d}x?$$

2. Let $1 \le p \le \infty$ and Ω be a bounded domain. Prove that if $u \in W^{1,p}(\Omega)$ then its modulus $|u|$, positive part $u^+ := u\,\mathbb{1}_{u>0}$ and negative part $u^- := -u\,\mathbb{1}_{u<0}$ belong to $W^{1,p}(\Omega)$. Find their derivatives in terms of Du.

   Hint: For $\varepsilon > 0$ consider function $F_\varepsilon(z) = (z^2+\varepsilon^2)^{1/2} - \varepsilon$. Prove that $F_\varepsilon(z)$ is Lipschitz with constant 1 and C^1 with $|F_\varepsilon'| \le 1$. Moreover $F_\varepsilon(z) \to |z|$ and $F_\varepsilon'(z) \to \mathrm{sgn}(z)\,\mathbb{1}_{z\neq 0}$ as $\varepsilon \to 0$. Similarly, consider function $G_\varepsilon(z) = \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if } z \ge 0, \\ 0 & \text{if } z < 0. \end{cases}$

Homework 9: problems for 13/05/2021

1. Watch the video on unbounded operators and solve the following problem. Let $(M, D(M))$ be an unbounded operator on $L^2(\mathbb{R})$:

$$(Mf)(x) = x\, f(x)$$

with the domain defined as

$$D(M) = \{\varphi \in L^2(\mathbb{R}) : x\, \varphi(x) \in L^2(\mathbb{R})\}.$$

Decide whether:

(A) M is bounded as an operator $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

(B) $(M, D(M))$ is densely defined,

(C) $(M, D(M))$ is closed.

Hint: Recall that when $f_n \to f$ in L^2, one may choose a subsequence converging a.e.

2. Prove that on $H^2_0(\Omega)$, the map $u \mapsto \|\Delta u\|_2$ defines an equivalent norm, i.e. there is a constant C (independent of u!) such that

$$\|\Delta u\|_2 \le \|u\|_{H^2}, \qquad \|u\|_{H^2} \le C\, \|\Delta u\|_2.$$

Hint: Use both Poincare inequality and smooth approximation. Observe that if u is smooth and compactly supported in Ω we have $\int_\Omega u_{x_i\, x_i}\, u_{x_j\, x_j} = \int_\Omega u_{x_i\, x_j}\, u_{x_i\, x_j}$.

Homework 10: problems for 20/05/2021

1. Let $u \in W_0^{1,p}(\Omega)$. Prove that the trivial extension $\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases}$ belongs to $W^{1,p}(\mathbb{R}^d)$ so that in this case we don't need extension theorem. *Hint*: Extend approximation of u.

2. This problem shows that all integration-by-parts formulas hold true for Sobolev functions if their boundary value is replaced with its trace. For example, prove that

$$\int_\Omega D_j u(x)\, v(x) + \int_\Omega u(x)\, D_j v(x) = \int_{\partial\Omega} (Tu)(x)\, (Tv)(x)\, n_j(x)$$

for $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, where Tu and Tv denotes traces of u and v, $1 < p < \infty$ and p' is the usual Holder conjugate.

This week's problems are easier but please take time to work on the second group project about Sobolev spaces.

Homework 11: problems for 27/05/2021

1. (A) Let $u \in W^{1,p}(\Omega)$ where Ω is bounded and connected domain. Prove that if u vanishes on $U \subset \Omega$ and $|U| > 0$ then

$$\|u\|_{L^p(\Omega)} \leq C(\Omega, U, p)\, \|Du\|_{L^p(\Omega)}.$$

Hint: Argue by contradiction (as in Problem 1, Homework 6) and use compact embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$.

(B) Deduce from (A) Poincare inequality for $u \in W_0^{1,p}(\Omega)$:

$$\|u\|_{L^p(\Omega)} \leq C(\Omega, p)\, \|Du\|_{L^p(\Omega)}.$$

We proved this before with integration by parts. *Hint:* Use extension from previous homework.

2. For this problem you may assume the following general Poincare inequality with averages: if $u \in W^{1,p}(\Omega)$ where Ω is bounded with C^1 boundary we have

$$\|u - (u)_\Omega)\|_{L^p(\Omega))} \leq C(\Omega, p)\, \|Du\|_{L^p(\Omega)},$$

where $(u)_\Omega$ is average of u over Ω:

$$(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)\ \mathrm{d}x.$$

This will be proved in the lecture using compact embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ as in the Problem 1(A) above.

(A) Prove explicit form of Poincare inequality for balls: if $u \in W^{1,p}(B(x,r))$

$$\|u - (u)_{B(x,r)}\|_{L^p(B(x,r))} \leq C(p)\, r\, \|Du\|_{L^p(B(x,r))}$$

and the constant C is independent of r.
Hint: Consider $v(y) = u(x + r\,y)$ and prove that $v \in W^{1,p}(B(0,1))$.

(B) Prove that if $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then u belongs to the space of functions of bounded mean oscillation (BMO), i.e.

$$|u|_{BMO} := \sup_{B(x,r) \subset \mathbb{R}^n} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u - (u)_{B(x,r)}| < \infty$$

This is Sobolev embedding for $p = n$.

Space BMO became absolutely fundamental in PDEs after John and Nirenberg proved celebrated John-Nirenberg inequality in the paper from 1961 published in CPAM. Among an others, it was used by Moser to solve XIX Hilbert's Problem.

Homework 12: problems for 10/06/2021

1. Let $f \in L^2(\Omega)$. We say that $u \in H_0^2(\Omega)$ is a weak solution to the biharmonic equation

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega$$

provided for all $\varphi \in H_0^2(\Omega)$ we have

$$\int_\Omega \Delta u \, \Delta\varphi = \int_\Omega f \, \varphi.$$

(A) Suppose that $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution to the biharmonic equation. Prove that it is also a weak solution. *Hint:* If $u \in H_0^2(\Omega)$ then $u_{x_i} \in H_0^1(\Omega)$.

(B) Prove that there exists the unique weak solution of biharmonic equation.

2. Let $\lambda \in \mathbb{R}$, $f \in L^2(\Omega)$ and consider equation

$$-\Delta u + \lambda \, u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{3}$$

(A) Define weak solutions in $H_0^1(\Omega)$

(B) For which λ one can use Lax-Milgram to obtain well-posedness of this problem in $H_0^1(\Omega)$?

(C*) One can prove that in fact the unique solution to (3) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$. Consider unbounded operator $A = \Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Prove that $(0, \infty)$ belongs to the resolvet of A. Moreover, if $R_\lambda : L^2(\Omega) \to L^2(\Omega)$ is resolvent operator we have an estimate

$$\|R_\lambda\| \le \frac{C}{\lambda} \qquad (\lambda > 0),$$

the constant C is independent of λ. This shows that A satisfies assumptions of Hille-Yosida theorem.