Jakub Łubiński, 394607 gr 1.

1

Zadanie 1

Pokażemy najpierw, że $\forall_{x_1,x_2\geq 0}\ x_1^p+x_2^p \leq (x_1^2+x_2^2)^{\frac{p}{2}}$, $p\geq 2$.

Dla $x_2=0$ nierówność jest oczywista. Dzieląc przez $x_2\neq 0$ stronami i podstawiając $t=\frac{x_1}{x_2}$ mamy $t^p+1\leq (t^2+1)^{\frac{p}{2}}$, $t\geq 0$.

Rozważmy funkcję $f(t)=(t^2+1)^{\frac{p}{2}}-t^p-1$. Wówczas $f(0)=0$, a $f'(t)=pt\left((t^2+1)^{\frac{p}{2}-1}-t^{p-2}\right)\geq 0$, a zatem $f(t)\geq 0$, dla $t\geq 0$, co chcieliśmy pokazać.

Wstawiając teraz do tej nierówności $x_1=\left|\frac{f(x)+g(x)}{2}\right|$ i $x_2=\left|\frac{f(x)-g(x)}{2}\right|$ dostajemy:

$$\forall_{x\in\Omega}\ \left|\frac{f(x)-g(x)}{2}\right|^p+\left|\frac{f(x)+g(x)}{2}\right|^p\leq\left(\frac{f(x)^2}{2}+\frac{g(x)^2}{2}\right)^{\frac{p}{2}}\leq\frac{1}{2}\left(|f(x)|^p+|g(x)|^p\right)$$

z nierówności Jensena dla funkcji $x\mapsto |x|^{\frac{p}{2}}$, która jest wypukła dla $p\geq 2$.

Skoro nierówność zachodzi $\forall_{x\in\Omega}$ to mamy:

$$\left\|\frac{f+g}{2}\right\|_p^p+\left\|\frac{f-g}{2}\right\|_p^p=\int_\Omega\left|\frac{f(x)-g(x)}{2}\right|^p+\left|\frac{f(x)+g(x)}{2}\right|^p d\mu(x)\leq\int_\Omega\frac{1}{2}\left(|f(x)|^p+|g(x)|^p\right)d\mu(x)=$$

$$=\frac{1}{2}\left(\|f\|_p^p+\|g\|_p^p\right).$$

Aby pokazać jednostajną wypukłość przestrzeni ustalmy $\varepsilon>0$ i weźmy $\|f\|_p=1$ i $\|g\|_p=1$, $\|f-g\|_p\geq\varepsilon$.

Wtedy z wykazanej nierówności:

$$\left\|\frac{f+g}{2}\right\|_p^p\leq 1-\left\|\frac{f-g}{2}\right\|_p^p\leq 1-\frac{\varepsilon^p}{2^p}\Rightarrow\left\|\frac{f+g}{2}\right\|_p\leq\left(1-\frac{\varepsilon^p}{2^p}\right)^{\frac{1}{p}}$$

Biorąc $\delta\in 1-\left(1-\frac{\varepsilon^p}{2^p}\right)^{\frac{1}{p}}$ mamy tezę.

Mariusz Janosz

2. $T: l^1 \to c_0^*$

$$(Ty)(x) = \sum_{i=1}^{\infty} x_i y_i$$

$$|T_y x| = |\sum x_i y_i| \leq \sum |x_i||y_i| \leq \sup|x_i| \cdot \sum |y_i| < \infty$$ ✓

bo $x_n \to 0$ więc $\sup|x_i| < \infty$ oraz $y \in l^1$

więc $\sum |y_i| < \infty$. Czyli T jest dobrze określony.

T jest różnowartościowy bo jeżeli

$y_1 \neq y_2$ to $\exists j \in \mathbb{N}$ taki, że $(y_1)_j \neq (y_2)_j$ różnią się na

j-tej współrzędnej

Wtedy $T_{y_1}(e_j) \neq T_{y_2}(e_j)$ gdzie $e_j = (0, \dots, 0, 1, 0, \dots)$ j-ty el. bazy Schaudera

Więc $T_{y_1} \neq T_{y_2}$ ✓

T jest na bo Niech $f \in (c_0)^*$ dowolny

weźmy $y = (f(e_1), f(e_2), \dots)$

~~Wówczas $\forall x \in c_0$, $x = \sum_{i=1}^{\infty} a_i e_i$~~ ~~$f(x) = f(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{N} a_i y_i + f(\sum_{i=N+1}^{\infty} a_i e_i)$~~

~~$\sum_{i=1}^{N} a_i y_i = f(\sum_{i=1}^{N} a_i e_i) \xrightarrow[N \to \infty]{} f(x)$~~

Weźmy $x_n = (\cancel{1, \dots, 1}, 0, \dots, 0)\ (\mathrm{sgn}(y_1), \dots, \mathrm{sgn}(y_n), 0, \dots, 0)$

~~jedynek~~

$\forall n \in \mathbb{N}$ $x_n \in c_0$ $\|x_n\| = 1$

$f \in (c_0)^*$ więc $\sup_{\|x\|=1} |f(x)| = \|f\| < \infty$

Więc $\{f(x_n)\}$ jest ograniczony

ale $\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} \mathrm{sgn}(y_i) \cdot y_i = f(x_n)$ zatem ciąg $\sum_{i=1}^{n} |y_i|$

ograniczony i niemalejący czyli zbieżny więc $y \in l^1$ ✓

Niech $x \in c_0$ $\quad x = \sum_{i=1}^{\infty} x_i e_i$

$$f(x) = f\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{N} x_i f(e_i) + f\left(\sum_{i=N+1}^{\infty} x_i e_i\right)$$

(*) Więc $\quad \sum_{i=1}^{N} x_i y_i = f\left(x - \sum_{i=N+1}^{\infty} x_i e_i\right) \xrightarrow[N\to\infty]{} f(x)$ (*)

Zatem $T_y x = f(x)$ czyli T jest na $(c_0)^*$.

$$\|y\|_{\ell^1} = \sum_{i=1}^{\infty} |y_i| = \left|\lim_{n\to\infty} T_y(\mathrm{sgn}(y_1), \dots, \mathrm{sgn}(y_n), 0, \dots)\right|$$

$$\forall_n \ \|(\mathrm{sgn}(y_1), \dots, \mathrm{sgn}(y_n), 0, \dots)\|_{\infty} = 1$$

Więc $\|y\|_{\ell^1} \leq \|T_y\|_{(c_0)^*}$

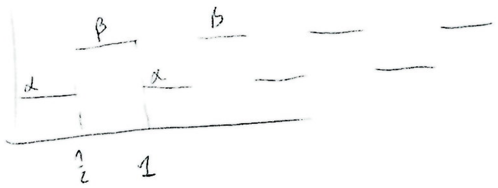
$$|T_y x| \leq \sum |y_i x_i| \leq \sup_{\|x\|=1} |x_i| \sum_{i=1}^{n} |y_i| \leq 1 \cdot \sum |y_i|$$

więc $\|T_y\|_{(c_0)^*} \leq \|y\|_{\ell^1}$ $\quad$ czyli $\|y\|_{\ell^1} = \|T_y\|_{(c_0)^*}$

Zatem T ustala izomorfizm $\underbrace{\text{pomiędzy}}_{\text{izometryczny}}$ ℓ^1 i $(c_0)^*$

bo T jest liniowy.

Zad 5 S. Antoniak 394197



1

$f_n(x) = f(nx)$

$f_n \xrightarrow{?} F$ in L^p

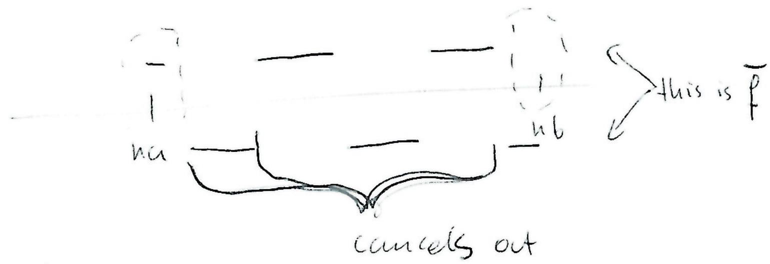
I claim it converges to $\frac{\alpha+\beta}{2} \equiv: F$. Considering that all $\varphi \in (L^p)^*$ are $\varphi(f) = \int f \cdot y$, $y \in L^q$

I will first show it for a step funtion, then a simple function, then I will use the fact that any $y \in L^q(0,1)$ can be approx. in L^q by a simple function.

1° y is a step function on $[a,b] \subset (0,1)$ $y = L \cdot \mathbb{1}_{[a,b]}$

$$\left| \int_0^1 f_n(x) y(x) dx - \int_0^1 F(x) y(x) dx \right| = \left| \int_0^1 (f_n - F)(x)\, y(x) dx \right| = \left\{ \begin{array}{l} t = nx \\ \frac{1}{n} dt = dx \end{array} \right\} = \frac{1}{n} \left| \int_0^n \underbrace{(\cancel{f_n} - F)(\cancel{x})}_{= \bar{f}}\, y\left(\tfrac{t}{n}\right) dt \right|$$

$$= \frac{1}{n} \left| \int_0^n \bar{f}(t)\, y\left(\tfrac{t}{n}\right) dt \right| = \frac{1}{n} \left| \int_{na}^{nb} L \cdot \bar{f}(t)\, dt \right|$$



$\leq \left(\frac{1}{n} \cdot L \cdot 3 \cdot \frac{1}{2} \cdot \frac{\alpha+\beta}{2} \right) \to 0$ (at most 3 full segments don't cancel out)
($\frac{1}{2}$ is the max length of them)

2° $y = \sum^k a_i S_i$

$$\left| \int_0^1 \dots - \int_0^1 \dots \right| = \frac{1}{n} \left| \int_0^n \bar{f}(t) \cdot \sum a_i S_i\left(\tfrac{t}{n}\right) \right| \leq \frac{1}{n} \sum^k |a_i| \left| \int_0^n \bar{f}(t)\, S_i\left(\tfrac{t}{n}\right) \right| \to 0$$ from ~~other~~ case 1°

finite number

3° $y \in L^q$ lets take Y simple that $\|y - Y\|_{L^q} < \varepsilon$

$$\left| \int_0^1 \bar{f}_n(x)\, y(x) dx \right| = \left| \int_0^1 \bar{f}_n(x)(y - Y)(x) + \int_0^1 \bar{f}_n(x) Y(x) \right| \leq \underbrace{\left| \int_0^1 \bar{f}_n(x)(y - Y)(x) dx \right|}_{(*)} + \underbrace{\left| \quad \right|}_{\to 0 \text{ from } 2^\circ}$$

$(*)$ by Hölder: $\left|\int_0^1 \quad\right| \leq \int_0^1 \left| \quad \right| \overset{H}{\leq} \underbrace{\|Y-y\|_q}_{<\varepsilon} \underbrace{\|\bar{f}_n\|_p}_{|\bar{f}_n| \leq \left|\frac{\alpha+\beta}{2}\right|} < \varepsilon \cdot \left|\frac{\alpha+\beta}{2}\right|$

2° $u \in L^p(\mathbb{R})$

$g_n(x) = u(x+n)$, g_n obviously is in L^p $\quad \|g_n\|_p = \|u\|_p$ $(*)$ (a simple variable substitution)

it converges weakly to $G \equiv 0$.

Since $u \in L^p(\mathbb{R})$, then $\forall_{\varepsilon>0} \exists M_\varepsilon$: $\left(\int_{\mathbb{R}\setminus[-M_\varepsilon, M_\varepsilon]} |u|^p\right)^{1/p} < \varepsilon$.

As before:

1° $y \in L^q$ is a step function on $[a,b]$

$$\left|\int_{\mathbb{R}} g_n \cdot y\right| = \left|\int_a^b g_n \cdot L\right| = \left|\int_{a+n}^{b+n} u \cdot L\right| \overset{\text{Höld}}{\leq} \int_{a+n}^{b+n} | \quad | \leq \underbrace{\|u\|_{p \,|\, \mathbb{R}\setminus[-M_\varepsilon, M_\varepsilon]}}_{<\varepsilon} \cdot \underbrace{\|L\|_{q \,|\, [a+n, b+n]}}_{< L\cdot(b-a)}$$

I take a bigger set

$< \varepsilon \cdot (L(b-a))$ when $a+n > M_\varepsilon$

2° y is simple $\quad y = \sum_{i=1}^k s_i a_i$, where s_i are step f.

$$\left|\int_{\mathbb{R}} g_n \cdot \sum^k s_i a_i\right| \leq \sum^k a_i \left|\int_{\mathbb{R}} g_n s_i\right| < k\,\varepsilon \cdot \sum a_i \text{ for sufficiently large } n \text{ by case } 1°$$

3° $y \in L^q$, $\|Y - y\|_q < \varepsilon_2$ $\quad$ Y is simple

$$\left|\int g_n \cdot y\right| \leq \underbrace{\left|\int g_n Y\right|}_{\substack{<\varepsilon \\ \text{from } 2°}} + \underbrace{\left|\int g_n (y-Y)\right|}_{\leq \int | \quad | \,\overset{\text{Höld}}{\leq}\, \|g_n\|_{L^p} \|y-Y\|_q}$$

$\|g_n\|_{L^p} \overset{(*)}{=} \|u\|_p$, $\quad \|y-Y\|_q < \varepsilon_2$

So in all 3 cases for any ε and $y \in L^q$ we find n sufficiently large, so that $\left|\int g_n \cdot y\right| < \varepsilon$.

3° $h_n(x)$ on $L^p(0,1)$ $\quad h_n(x) = e^{-nx} \cdot n^{1/p}$.

- for a given x $\quad \lim_{n\to\infty} h_n(x) = \frac{n^{1/p}}{e^{nx}} \leq \frac{n}{e^{nx}} \to 0$ as exp increases asymptotically faster (AM 1) than a polynomial

EXCEPT for $x=0$, then $h_n(x) = n^{1/p} \to \infty$

5 S. Antonich continuation

- does $h_n \to 0$?

let's observe that on $[a,b] \subset (0,1)$ h_n is decreasing on $[a,b]$ and is maximal at a. Thus, since $h_n(a) \to 0$, then $h_n \rightrightarrows 0$ on $[a,b]$

$|h_n| < \varepsilon$ for large n

As before

1° $y \in L^q$ a step function $L \cdot \mathbb{1}_{[a,b]}$

$$\left| \int_0^1 y\, h_n \right| = \left| L \cdot \int_a^b h_n \right| \le |L| \left| \int_a^b h_n \right| \le |L| \int_a^b |h_n| \le |L| \int_a^b \varepsilon \le L(b-a)\varepsilon$$

for large n

2° $y \in L^q$ is simple

$$\left| \int_0^1 h_n \cdot y \right| = \left| \int_0^1 h_n \cdot \sum_{i=1}^k a_i s_i \right| \le \sum_{i=1}^k a_i \left| \int_0^1 h_n \cdot s_i \right| < \sum_{i=1}^k a_i\, \varepsilon$$

for large n

since there is a finite number of step functions, so we use 1°

3° $\|y - Y\|_q < \varepsilon_2$, Y is simple, $y \in L^q(0,1)$

$$\left| \int_0^1 h_n \cdot y \right| \le \underbrace{\left| \int h_n Y \right|}_{\to 0 \text{ from } 2^\circ} + \left| \int h_n (y - Y) \right|$$

$$\le \int |h_n (y-Y)| \overset{\text{Höld}}{\le} \underbrace{\|h_n\|_p}_{\text{bounded?}} \underbrace{\|y - Y\|_q}_{< \varepsilon_2}$$

and
$$\|h_n\|_p^p = \int_0^1 n\, e^{-npx}\, dx = n \cdot \frac{1}{-np} \cdot e^{-npx} \Big|_0^1 = \left| -\frac{1}{p} \cdot \left(e^{-np} - 1\right) \right| = \frac{1 - e^{-np}}{1} < 2$$

$\downarrow$

$1 \neq 0$

So $\|h_n\|_p^p$ is bounded which completes 3°

and $h_n \overset{L^p}{\not\to} 0$.