Comment on Big Homework 3

Problem 3 This is standard problem for Banach-Steinhaus Theorem (pointwise properties "for each $f \in E^*$" $\Rightarrow$ global properties).
However, this needs formula $\|x\| = \sup_{f \in E^*, \|f\| \le 1} f(x)$ known as a consequence of Hahn-Banach theorem.

↑ This supremum is attained

For each $x \in A$ we define $T_x(f) = f(x)$, $T_x : E^* \to \mathbb{R}$.
Then, if we fix $f \in E^*$, $\sup_{x \in A} |T_x(f)| = \sup_{x \in A} |f(x)| < \infty$ as $f(A)$ is bounded.
By BS we have
$$\underbrace{\sup_{x \in A} \sup_{f \in E^*, \|f\| \le 1} |T_x(f)|}_{\|} \le C \quad \text{for some constant}$$
$$\sup_{x \in A} \underbrace{\sup_{f \in E^*, \|f\| \le 1} |f(x)|}_{\|x\|} \le C$$
$\square$

Remark: In Special Problem 4 we learn that it is not true in general that supremum in $\|f\|_{E^*} = \sup_{\|x\| \le 1} |f(x)|$ is attained but it is true for reflexive spaces.

Problem 4: This was IMPORTANT problem. In Analysis, we sometimes have to work in setting with less regular structure (not Banach or normed space). Still, one can sometimes use some embedding to make the setting regular enough.

Here, we use $F(x) = (\varphi_1(x), \ldots, \varphi_k(x), \varphi(x))$ and recall that $\forall_i\ \varphi_i(v) = 0 \Rightarrow \varphi(v) = 0$.

First, we note that $(0,0,\dots,0,1) \notin \operatorname{Im} F \subset \mathbb{R}^{n+1}$. We want to separate $\operatorname{Im} F$ and $(0,0,\dots,0,1)$.

closed as linear subspace in $\mathbb{R}^{n+1} \to$ finite dim.

By Hahn-Banach (geometric with closed and compact sets), there is $\xi \in (\mathbb{R}^{n+1})^*$, $\lambda \in \mathbb{R}$ s.t.

$$\xi((0,0,\dots,0,1)) < \lambda < \xi(\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x), \varphi(x)) \quad \forall x \in X$$

But $\xi \in (\mathbb{R}^{n+1})^*$ so it has the form $\xi(\underset{\mathbb{R}^{n+1}}{\underset{\cap}{y}}) = \sum_{i=1}^{n+1} \lambda_i y_i$. Therefore

$$\lambda_{n+1} < \lambda < \sum_{i=1}^{n} \lambda_i \varphi_i(x) + \lambda_{n+1} \varphi(x) \qquad (*)$$

<u>Claim:</u> $\sum_{i=1}^{n} \lambda_i \varphi_i(x) + \lambda_{n+1}\varphi(x) = 0 \quad \forall x \in X$. For if not, we can always scale x so that $(*)$ is not satisfied.

Therefore $(*)$ gives $\lambda_{n+1} < \lambda < 0$ so $\lambda_{n+1} \neq 0$ and

$$\varphi(x) = \sum_{i=1}^{n} \left(-\frac{\lambda_i}{\lambda_{n+1}}\right) \varphi_i(x)$$

as desired ;)

$\square$.