Comment on Christmas Problems:

(1) Our definition of orthonormal basis in Hilbert space $(H, \langle \cdot, \cdot \rangle)$ was

(A) $\begin{cases} \bullet \ \{e_i\}_{i=1}^{\infty} \text{ such that } \|e_i\| = 1, \ \langle e_i, e_j \rangle = 0 \quad i \neq j \\ \bullet \text{ for any } x \in H, \ x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \text{ and the series is convergent in } H. \end{cases}$

The following definition was used in the lecture

(B) $\begin{cases} \bullet \ \{e_i\}_{i=1}^{\infty} \text{ s.t. } \|e_i\| = 1, \ \langle e_i, e_j \rangle = 0 \text{ for } i \neq j \\ \bullet \text{ if } \langle x, e_i \rangle = 0 \ \forall_i \Rightarrow x = 0. \end{cases}$

The target of this exercise was to verify that $B \Rightarrow A$ (implication $A \Rightarrow B$ is trivial). So we get two equivalent definitions. Any of them implies Parseval identity, that is

$$\|x\|^2 = \sum_{i=1}^{\infty} (x, e_i)^2$$

(from Bessel's inequality we only know $\|x\|^2 \geq \sum_{i=1}^{\infty} (x, e_i)^2$).
This is used to solve part 1C).

(2) Operator in Problem 2 can be considered as "multiplication operator on ℓ^2". Note that this result (multiplication is compact for $a \in c_0$) is very different from multiplication on $L^2(\mathbb{R}^d)$. We have seen cf. Problem C10 in PS9 that only multiplying by 0 can be compact on $L^2(\mathbb{R}^d)$.