Functional Analysis (WS 19/20), Christmas Problems

deadline: 9/01/2020

Important: If you submit the solutions (envelope - room 4040) at most on 8/01/2020, they will be corrected before 9/01/2020.

1. This is a revision exercise on orthonormal basis in Hilbert spaces. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal Schauder basis of Hilbert space H defined as in the lecture:

   (A) $\{e_i\}_{i=1}^{\infty}$ is an orthonormal system, i.e. $\|e_i\| = 1$ and $\langle e_i, e_j \rangle = 0$ for $i \neq j$,

   (B) it is complete i.e. for any $x \in H$ if $\langle x, e_i \rangle = 0$ for all i then $x = 0$.

   Prove that:

   (a) For any $x \in H$, the series $\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ converges in H and the limit is x. *Remark:* Recall that this was our definition of orthonormal Schauder basis presented in the tutorial. *Hint:* Use completeness.

   (b) For any $y \in l^\infty$, the sequence $\frac{1}{n} \sum_{k=1}^{n} y_k e_k$ converges to 0 in H.

   (c) Compute $\sum_{k \in \mathbb{Z}} \left| \int_0^1 x^2 e^{ikx} dx \right|^2$.

2. (a) Let $a = (a_n)_{n \in \mathbb{N}} \in l^\infty$ and e_n be a standard Schauder basis of l^p with $1 \leq p < \infty$, i.e. $e_i = (0, ...0, 1, 0, 0, ...)$ with 1 on the i-th position. We define map T on l^p with $Te_n = a_n e_n$ so T is a priori defined on span$(e_1, e_2, ...)$ (finite linear combinations of e_i). Prove that T can be uniquely extended to a bounded operator $T : l^p \to l^p$. By a slight abuse of notation, we still denote this extension with T. Write a formula for T and justify it.

   (b) Find all bounded sequences a as above such that the operator $T : l^p \to l^p$ for $1 \leq p < \infty$ is compact.