Michał Raszkowski PD AF na 09.10.2019

L1) $f_i \in L^{p_i}$, $i=1,\dots,n$, $\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p}$

Teza: $\|f_1 \dots f_n\|_p \leq \|f_1\|_{p_1} \dots \|f_n\|_{p_n}$ (teza $\Rightarrow$ $f_1 \dots f_n \in L^p$, bo $f_i \in L^{p_i}$)

Dowód: indukcja po n. B.s.o. $p_1 \leq \dots \leq p_n$

$n=1$: $\|f_1\|_p \leq \|f_1\|_{p_1}$, bo $\frac{1}{p} = \frac{1}{p_1} \Leftrightarrow p = p_1$

Założmy prawdziwość dla $n-1$. Rozważmy dwa przypadki:

1) $p_n = \infty \Rightarrow \sum_{i=1}^{n-1} \frac{1}{p_i} = \frac{1}{p}$

$$\|f_1 \dots f_n\|_p \underset{\text{wyciągam supremum } f_n}{\leq} \|f_1 \dots f_{n-1}\|_p \|f_n\|_\infty \underset{\text{zał. ind.}}{\leq} \|f_1\|_{p_1} \dots \|f_{n-1}\|_{p_{n-1}} \underbrace{\|f_n\|_\infty}_{= \|f_n\|_{p_n}}$$

2) $p_n < \infty \Rightarrow p < \infty$ (bo $p_1 \leq \dots \leq p_n$)

Niech $q_1 = \frac{p_n}{p_n - p}$, $q_2 = \frac{p_n}{p}$. $q_1, q_2 \in (1, \infty)$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$ zatem z nier. Hölder'a

$$\| |f_1 \dots f_{n-1}|^p |f_n|^p \|_1 \leq \| |f_1 \dots f_{n-1}|^p \|_{q_1} \| |f_n|^p \|_{q_2}$$

$$\Updownarrow$$

$$\int |f_1 \dots f_n|^p \leq \left(\int |f_1 \dots f_{n-1}|^{pq_1}\right)^{\frac{1}{q_1}} \left(\int |f_n|^{pq_2}\right)^{\frac{1}{q_2}}$$

$$\Updownarrow$$

$$\left(\int |f_1 \dots f_n|^p\right)^{\frac{1}{p}} \leq \left(\int |f_1 \dots f_{n-1}|^{pq_1}\right)^{\frac{1}{pq_1}} \left(\int |f_n|^{pq_2}\right)^{\frac{1}{pq_2}}$$

$\Downarrow$ zał. ind. (bo $\sum_{i=1}^{n-1} \frac{1}{p_i} = \frac{1}{p} - \frac{1}{p_n} = \frac{1}{p}\left(1 - \frac{p}{p_n}\right) = \frac{1}{pq_1}$) oraz $pq_2 = p_n$

$$\|f_1 \dots f_n\|_p \leq \|f_1\|_{p_1} \dots \|f_{n-1}\|_{p_{n-1}} \cdot \|f_n\|_{p_n}$$

□

(1/1)

L6 – Tomasz Przybyłowski

W.l.o.g. we may assume $r \in (p, q)$.

Firstly, let $q < \infty$. In this case $\frac{1}{r} \in \left(\frac{1}{q}, \frac{1}{p}\right)$, so $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ for some $\alpha \in (0,1)$.

Then:

$$\infty > \|f\|_p^\alpha \cdot \|f\|_q^{1-\alpha} = \left(\int |f|^p\right)^{\frac{\alpha}{p}} \cdot \left(\int |f|^q\right)^{\frac{1-\alpha}{q}} = \left(\left(\int |f|^p\right)^{\frac{r\alpha}{p}} \cdot \left(\int |f|^q\right)^{\frac{r(1-\alpha)}{q}}\right)^{\frac{1}{r}}$$

Hölder for $p_1 = \frac{p}{r\alpha}, q_1 = \frac{q}{r(1-\alpha)}$

$$\geq \left(\int |f|^{r\alpha} \cdot |f|^{r(1-\alpha)}\right)^{\frac{1}{r}} = \left(\int |f|^r\right)^{\frac{1}{r}} = \|f\|_r,$$

so ~~więc~~ $f \in L^r$.

Now, let $q = \infty$. This means there is $m > 0$ s.t. $|f| \leq m$ almost everywhere.

To prove $f \in L^r$ it's enough to prove that $\int |f|^r < \infty$. Let's go:

$$\int |f|^r = m^r \cdot \int \left|\frac{f(x)}{m}\right|^r dx \overset{p \leq r \text{ and } \frac{f(x)}{m} \leq 1 \text{ a.e.}}{\leq} m^r \cdot \int \left|\frac{f(x)}{m}\right|^p dx = m^{r-p} \cdot \int |f|^p =$$

$$= m^{r-p} \cdot \|f\|_p^p < \infty,$$

so $f \in L^r$, which finishes the proof.

one can take inf over m s.t. $|f| \leq m$ a.e, and prove inequality $\|f\|_r \leq \|f\|_p^\alpha \|f\|_\infty^{1-\alpha}$

1/1

Mariusz Janosz

Z3.

$\mathcal{P}$ - wielomiany na $[0,1]$

$\|f\| = \sup_{x\in[0,1]} |f(x)|$

Niech $f(x) = 0 \ \forall x \in [0,1]$

$\Rightarrow \|f\| = \sup_{x\in[0,1]} |f(x)| = 0$

Niech $g \in \mathcal{P}$ taki, że $\|g\| = 0$

$\Rightarrow \|g\| = \sup_{x\in[0,1]} |g(x)| = 0 \Rightarrow \forall x \in [0,1] \ |g(x)| \leq 0$

ale $|g(x)| \geq 0 \ \forall x \in [0,1]$ więc $g \equiv 0$

czyli $\|*\| = 0 \Leftrightarrow * \equiv 0$.

Niech $a \in$ ~~ciało~~ $\mathbb{R}, (\mathbb{C})$ $\quad f \in \mathcal{P}$

$\|af\| = \sup_{x\in[0,1]} |a \cdot f(x)| = \sup_{x\in[0,1]} |a| \cdot |f(x)| = |a| \cdot \sup_{x\in[0,1]} |f(x)| = |a| \cdot \|f\|$

Niech $f, g \in \mathcal{P}$

$\|f+g\| = \sup_{x\in[0,1]} |f(x)+g(x)| = |f(x_0)+g(x_0)| \leq |f(x_0)| + |g(x_0)| \leq \sup|f| + \sup|g| = \|f\| + \|g\|$

sup funkcji ciągłej na zbiorze zwartym jest przyjmowane

zwykła nier. Δ

Więc $(\mathcal{P}, \|\cdot\|)$ jest przestrzenią unormowaną

ale nie jest przestrzenią Banacha bo

ciąg $f_n(x) = \sum_{k=0}^{n} \frac{(x)^k}{k!}$ $\quad f_n(x) \to \exp(x)$

jest ciągiem Cauchy'ego ale jego granica nie należy do $\mathcal{P}$.

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