Comments to the homework for 17/10/2019

Problem S4: c_0 is a Banach space

That was an exercise for choosing parameters like in standard $\epsilon - \delta$ arguments. For those of you still struggling with this type of proofs, I strongly recommend reading Kuba Woźnicki's homework and solution to Problem S3 (c is a Banach space) - both are available on our website. It may be also a good idea to try to solve these problems again some time later.

When one uses $\epsilon - \delta$ arguments, one has to choose some parameters (for instance, for all ϵ we choose some δ). We have to be extremely careful about possible dependence of these parameters (in this example, δ depends on ϵ and so, whenever one changes ϵ, δ will be also different). Some people stress it by writing $\delta(\epsilon)$. You can see that in Kuba's homework too.

Another issues some of you have faced is dealing with notation concerning sequences of sequences. It is a good practise to explain your notation at the beginning, say, $\{x^k\}$ is a sequence in c_0 and $x^k = (x_1^k, x_2^k, ...)$ so that the meaning of upper and lower indices is clear.

Below I have typed a detailed proof. To be clear - I didn't expect this type of precision but I believe (after reading your solutions) that it may be helpful for some of you to see all details.

First observation is that we know that c is a Banach space with $\|\cdot\|_\infty$ and $c_0 \subset c$. So in order to check that c_0 (with norm $\|\cdot\|_\infty$) is a Banach space, it is sufficient to verify it is closed in c with respect to $\|\cdot\|_\infty$ norm (Problem L4 in PS1). So let $\{x^k\} \subset c_0$ be a sequence converging to some $x \in c$ and we have to verify that $x \in c_0$, i.e. that $\lim_{n\to\infty} x_n = 0$.

Property $x^k \to x$ with respect to the norm $\|\cdot\|_\infty$ means that $\sup_n |x_n^k - x_n| \to 0$ as $k \to \infty$. In particular,

$$\forall_{\epsilon>0} \exists_{K(\epsilon)\in\mathbb{N}} \forall_{k\geq K(\epsilon)} \forall_{n\in\mathbb{N}} \quad |x_n^k - x_n| \leq \epsilon.$$

Aiming at proving $\lim_{n\to\infty} x_n = 0$, we estimate x_n. To this end, fix $\epsilon > 0$. Then, for all $n \in \mathbb{N}$ and all $k \geq K(\epsilon)$

$$|x_n| \leq |x_n^k - x_n| + |x_n^k| \leq \epsilon + |x_n^k|.$$

Again, as above inequality hold for all $n \in \mathbb{N}$[1] we can estimate $\limsup_{n\to\infty} |x_n|$:

$$\limsup_{n\to\infty} |x_n| \leq \epsilon + \limsup_{n\to\infty} |x_n^k| = \epsilon$$

as $\limsup_{n\to\infty} |x_n^k| = 0$ since $x^k \in c_0$ for all $k \in \mathbb{N}$ (in particular: for $k \geq K(\epsilon)$ so we can use it). Since ϵ is arbitrary (it was independently chosen at the beginning), we conclude $\limsup_{n\to\infty} |x_n| = 0$ so $\lim_{n\to\infty} x_n = 0$ as desired.

[1] It would be sufficient to know this for large n.

Problem S7: set l^1 with l^∞ norm is not a Banach space

We have seen similar problems (for instance L5 or C4 from PS1). Most of you had good intuition that l^1 with l^∞ norm is not a Banach space as the norm is not "well fitted" to the definition of the set[2] but not every justification was correct.

The simplest argument goes as follows. Suppose it is a Banach space. Then, every Cauchy sequence with respect to $\|\cdot\|_\infty$ norm is convergent in the set l^1. Note that convergent sequences are always Cauchy. Therefore every sequence convergent with respect to $\|\cdot\|_\infty$ norm is convergent in the set l^1. This means that if $\|x^k - x\|_\infty \to 0$ as $k \to \infty$, then $x \in l^1$.

One can consider sequence $x^k = \left(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k}, 0, 0, ...\right)$. This sequence converges in l^∞ to $x = \left(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k}, \frac{1}{k+1}, ...\right)$ as

$$\|x - x^k\|_\infty \leq \sup_{n \geq k+1} \frac{1}{n} \leq \frac{1}{k+1} \to 0 \text{ as } k \to \infty$$

(alternatively: one can deduce this directly from Problem S5 from PS1 as Schauder projections approximate sequences from c_0 in norm). Unfortunately, $x \notin l^1$ and the proof is concluded.

Some of the sequences from your solutions were incorrect. Let me show an example. Consider sequence $x^k = \left(1 - \frac{1}{k}, \left(1 - \frac{1}{k}\right)^2, \left(1 - \frac{1}{k}\right)^3, ...\right)$. It converges pointwisely (in each term separately) to $x = (1, 1, 1, ...)$ but not in l^∞. Indeed,

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \left| 1 - \left(1 - \frac{1}{k}\right)^n \right| \geq \lim_{k\to\infty} \left| 1 - \left(1 - \frac{1}{k}\right)^k \right| = 1 - \frac{1}{e} > 0.$$

Another issue was that some of the sequences were hard-to-analyze (at least for me) and you did not provide any details[3]. For instance, $x^k = (x_n)^k = n^{-1-\frac{1}{k+1}}$. We want to prove that x^k converges to the harmonic sequence i.e.

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \left| n^{-1} - n^{-1-\frac{1}{k+1}} \right| = 0.$$

One can consider function $f_k(x) = x^{-1}\left(1 - x^{-\frac{1}{k+1}}\right)$ and study its derivative to see that for nonnegative arguments, $f_k'(x) \geq 0$ if and only if

$$\left(1 + \frac{1}{k+1}\right)^{k+1} \geq x.$$

Therefore, if $x \leq e \leq 3$, $|f_k(n)| \leq 3$ and this holds for all $k \in \mathbb{N}$. Therefore

$$\sup_{n\in\mathbb{N}} \left| n^{-1} - n^{-1-\frac{1}{k+1}} \right| \leq \max(f_k(1), f_k(2), f_k(3)) \to 0 \text{ as } k \to \infty.$$

[2]In the nearest future, this intuition will be rigorously stated in terms of Inverse Mapping Theorem.

[3]If there is some simple proof I would be happy to see that.