Tomasz Przybyłowski - H13

Since M is a strictly contained closed subspace of X, then there exists $v \in X$ s.t. $\operatorname{dist}(v, M) > 0$. Let $\alpha := 1-\varepsilon$ for some $\varepsilon \in (0,1)$.

By the definition of dist, there exists $m_v \in M$, s.t.

$$\|v - m_v\| \in [\operatorname{dist}(v,M), (1+\varepsilon)\operatorname{dist}(v,M)] \qquad (*)$$

We claim that $x := \frac{v-m_v}{\|v-m_v\|}$ satisfies the statement. Indeed, let $u \in M$.

Then

$$\left\| \frac{v-m_v}{\|v-m_v\|} - u \right\| = \frac{1}{\|v-m_v\|} \cdot \| v - \underbrace{(m_v + u \cdot \|v-m_v\|)}_{\in M} \| \geq \frac{\operatorname{dist}(v,M)}{\|v-m_v\|} \overset{(*)}{\geq} \frac{\operatorname{dist}(v,M)}{(1+\varepsilon)\operatorname{dist}(v,M)} =$$

$$= \frac{1}{1+\varepsilon} \geq 1-\varepsilon = \alpha,$$

so for each $u \in M$ we have: $\|x-u\| \geq \alpha$, which means $\operatorname{dist}(x, M) \geq \alpha$, as desired. ∎

OK (1/1)