Functional Analysis (WS 19/20), Problem Set 1

(normed spaces, Banach spaces, L^p spaces)

L^p spaces

We write L^p for $L^p(X, \mathcal{F}, \mu)$ where $\mathcal{F}$ is a σ-algebra and μ is a σ-finite measure on $(X, \mathcal{F})$. We write $\|\cdot\|_p$ for L^p norm.

L1. (**generalized Hölder inequality**) Let $f_i \in L^{p_i}$ for $i = 1, ..., n$ where $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$. Prove that $f_1 f_2 ... f_n \in L^p$. More precisely, prove the bound

$$\|f_1 f_2 ... f_n\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} ... \|f_n\|_{p_n}.$$

Hint: one can simplify to the case $p = 1$, then proceed by induction.

L2. (**generalized Minkowski inequality**) Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be two measure spaces. Let $F : X \times Y \to \mathbb{R}$ be a measurable and nonnegative map. Prove that

$$\left| \int_Y \left| \int_X F(x,y) d\mu(x) \right|^p d\nu(y) \right|^{\frac{1}{p}} \leq \int_X \left| \int_Y |F(x,y)|^p d\nu(y) \right|^{\frac{1}{p}} d\mu(x).$$

Deduce from this standard Minkowski inequality.

L3. Suppose $\mu(X) < \infty$. Check that $L^p \subset L^q$ whenever $p \geq q$. Prove that assumption $\mu(X) < \infty$ is necessary.

L4. Let $(X, \|\cdot\|_X)$ be a Banach space and $Y \subset X$ a subset of X. Prove that $(Y, \|\cdot\|_X)$ is a Banach space if and only if Y is closed in X.

L5. Consider set $L^2(0,1)$ equipped with $\|\cdot\|_1$ norm. Is it a normed space? Is it a Banach space? More generally, replace $L^2(0,1)$ with $L^p(0,1)$ where $1 < p \leq \infty$.

L6. (**Littlewood's inequality**) Let $f \in L^p \cap L^q$ for some $1 \leq p, q \leq \infty$. Prove that $f \in L^r$ for any $r \in [p,q]$. *Hint:* let $\alpha \in [0,1]$ and write $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. Then prove $\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha}$.

L7. Let $f \in L^p$. Prove that for any p_0 and p_1 such that $1 \leq p_0 < p$ and $p < p_1 < \infty$ there are $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ such that $f = f_0 + f_1$. Thus, f can be always decomposed for "better" and "worse" part. *Hint:* truncate f on your favourite level.

L8. Prove that a normed space $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series (i.e. $\sum_{k=1}^\infty \|x_k\| < \infty$) is convergent in X (i.e. sequence of partial sums is convergent in X).

L9. Use Exercise L8. to deduce that $(L^p, \|\cdot\|_p)$ (for $1 \leq p < \infty$) is a Banach space once again.

Spaces of continuous and differentiable functions

C1. Prove that space $C_0(\mathbb{R}^d)$ of continuous functions "vanishing at infinity" (i.e. $f(x) \to 0$ whenever $|x| \to \infty$) equipped with the supremum norm is a Banach space.

C2. Prove that space $C^1([0,1])$ of functions continuously differentiable on $[0,1]$, equipped with the norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$ is a Banach space.

C3. Let $\mathcal{P}$ be the space of all polynomials on $[0,1]$ equipped with supremum norm. Prove that $\mathcal{P}$ is a normed space but it is not a Banach space. Is $\mathcal{P}$ closed in $C([0,1])$? *Hint:* Use Problem L8. *Remark:* This Problem can be generalized in Problem A3.

C4. Are maps (a) $\|f\|_A := \sup_{x\in[0,1]} |f(x)|$ and (b) $\|f\|_B := \sup_{x\in[0,1]} |f'(x)|$ norms on $C^1([0,1])$? Do they form Banach spaces?

C5. Are maps

$$\|f\|_C := f(0) + \sup_{x\in[0,1]} |f'(x)|$$

and

$$\|f\|_D := \left(\int_0^1 (f(x))^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 (f'(x))^2 dx\right)^{\frac{1}{2}}$$

norms on $C^1([0,1])$? Do they form Banach spaces?

Spaces of sequences

S1. For $1 \le p < \infty$ we define l^p as the space of complex-valued (or real-valued if one has troubles with complex numbers) sequences summable with p-th power and equipped with the norm

$$\left\|(x_k)_{k=1}^{\infty}\right\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$$

For $p = \infty$, we define l^∞ as complex-valued bounded sequences with the norm

$$\left\|(x_k)_{k=1}^{\infty}\right\|_p = \sup_{k\in\mathbb{N}} |x_k|.$$

Justify briefly that l^p is a Banach space.

S2. (**Schauder basis for l^p**) Consider l^p where $1 \le p < \infty$ and unit vectors $e_i = (0,0,...,0,1,0,...)$ where 1 is on the i-th coordinate. Prove that for any $x = (x_1, x_2, ...) \in l_p$, $\sum_{i=1}^n x_i e_i \to x$ converges in l_p, i.e. that

$$\left\|\sum_{i=1}^n x_i e_i - x\right\|_p \to 0.$$

We say that system $\{e_i\}_{i\in\mathbb{N}}$ is a Schauder basis of l_p. How the situation changes for $p = \infty$?

S3. Consider space of real-valued sequences $(x_0, x_1, x_2, ...)$ such that $\lim_{k\to\infty} x_k$ exists and equip it with a supremum norm, i.e. $\|(x_k)_{k=1}^{\infty}\|_\infty = \sup_k |x_k|$. Prove that this is a Banach space (it is usually denoted with c).

S4. Similarly, consider subspace of c of sequences $(x_0, x_1, x_2, ...)$ converging to 0 equipped with supremum norm (it is usually denoted with c_0). Prove that it is a Banach space.

S5. (**Schauder basis for c_0**) Consider problem S2. with space c. More precisely, given unit vectors $e_i = (0,0,...,0,1,0,...)$ where 1 is on the i-th coordinate, prove that for any $x = (x_1, x_2, ...) \in c_0$, $\sum_{i=1}^n x_i e_i \to x$ converges in c_0, i.e. that

$$\left\|\sum_{i=1}^n x_i e_i - x\right\|_\infty \to 0.$$

Note once again, that according to Exercise S2., this is not the case for l^∞ but c_0 is a closed subset of l^∞.

S6. Consider subset of c of sequences converging to 1. Can this subset be a normed space (no matter how the norm is defined)?

S7. Consider set l^1 with l^∞ norm. Is it a normed space? Is it a Banach space?

S8. Consider set $\mathcal{A}$ of sequences $(x_n)_{n=1}^\infty$ such that $x_{2k} = \frac{1}{2}x_{2(k-1)}$ for all $k \in \mathbb{N}$. For which Banach spaces $(X, \|\cdot\|_X)$ among $(c_0, \|\cdot\|_\infty)$ and $(l_p, \|\cdot\|_p)$ where $1 \le p \le \infty$, set $\mathcal{A}$ is a closed subspace of $(X, \|\cdot\|_X)$ and therefore, it forms Banach space with norm $\|\cdot\|_X$?

Additional important problems

A1. (**Minkowski functional**) Let E be a normed space and $C \subset E$ be an open and convex subset with $0 \in C$. For every $x \in E$ we define:

$$\rho(x) = \inf\left\{\alpha > 0 : \frac{x}{\alpha} \in C\right\}.$$

Prove the following properties of ρ which is called Minkowski functional of C:

(a) $\rho(\lambda x) = \lambda\rho(x)$ for any $\lambda > 0$ and $x \in E$,

(b) $\rho(x+y) \le \rho(x) + \rho(y)$ for any $x, y \in E$,

(c) there is a constant M so that $0 \le \rho(x) \le M\|x\|$ for all $x \in E$,

(d) $C = \{x \in E : \rho(x) < 1\}$.

A2. (**Orlicz spaces**) Let $m : [0,\infty) \to [0,\infty)$ be a convex function with $m(0) = 0$. Prove that Orlicz space:

$$L^m(X) = \left\{f : X \to \mathbb{R} \text{ measurable and } \int_X m\left(\frac{|f(x)|}{\lambda}\right) dx < \infty \text{ for some } \lambda > 0\right\}$$

equipped with the so-called Luxemburg norm

$$\|f\|_{L^m} = \inf\left\{\lambda > 0 : \int_X m\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

is a normed space. Check that for $m(t) = t^p$, Orlicz spaces coincide with the classical L^p spaces.

A3. (**Banach spaces have uncountable basis**) More generally than in Exercise C3., prove that infinite dimensional Banach space has uncountable **Hamel** basis. Compare with C3. *Hint:* aiming at contradiciton, suppose there is a countable basis and apply Baire Cathegory Theorem.