Functional Analysis (WS 19/20), Problem Set 10

(convolutions and Schwartz spaces)

For $f, g \in L^1(\mathbb{R}^d)$ we define convolution $f * g$:

$$f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

Clearly $f*g = g*f$. Convolutions are also studied for functions defined on subsets $\Omega \subset \mathbb{R}^d$ assuming one takes care about appropriate domain of definition for f and g.

Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of infinitely differentiable functions such that the family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty$$

where $\alpha, \beta \in \mathbb{N}^d$. We usually say that Schwartz space consists of functions vanishing faster than any polynomial.

We say that $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ if $p_{\alpha,\beta}(f_n - f) \to 0$ as $n \to \infty$ for all $\alpha, \beta \in \mathbb{N}^d$.

Convolutions

C1. Let $f \in L^1(\mathbb{R}^n)$ and g be Lipschitz function. Prove that $f * g$ is again Lipschitz.

C2. Let $g \in C_0^k(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$. Prove that $f * g$ is $C^k(\mathbb{R}^n)$. Find the formulas for the derivatives of $f * g$.

C3. (**Young's inequality**) Use Riesz–Thorin Theorem to prove Young's convolutional inequality: if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ then $f*g \in L^r(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, 1 \le p, q, r \le \infty$. Moreover,

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Riesz–Thorin Complex Interpolation Theorem: Let $1 \le p_0, p_1, q_0, q_1 \le \infty$. Let T be a bounded operator from L^{p_0} to L^{q_0} as well as from L^{p_1} to L^{q_1}. Then, T is also a bounded operator from L^{p_θ} to L^{q_θ} where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \theta \in [0,1].$$

C4. Prove Young's convolutional inequality directly from the Hölder inequality.

C5. (**cutoff**) We fix a smooth radial function $\eta \in C^\infty$ supported on the ball $B_1(0)$ such that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$. Existence of such function can be assumed. We also consider scalings of η: $\eta_\epsilon = \epsilon^{-d}\eta(x/\epsilon)$. Note that η_ϵ is supported on $B_\epsilon(0)$ and $\int_{\mathbb{R}^n} \eta_\epsilon(x)\,dx = 1$. Prove that there is a smooth function f such that $f = 1$ on $B_1(0)$, $f = 0$ on $\mathbb{R}^n \setminus B_2(0)$ and $f \in [0,1]$. [1]

C6. Are smooth functions dense in $L^\infty(0,1)$?

[1] Function η is usually called a **mollifier**. Mollifiers are used to prove various results on density of smooth functions in L^p, $1 \le p < \infty$ as it was discussed in lectures of Analysis II. For instance, one can get that C_0^∞ is dense in L^p. For some review, see Theorem 4.1 in L.C.Evans, M.Gariepy, *Measure theory and fine properties of functions*.

Schwartz space

S1. Equip $\mathcal{S}(\mathbb{R}^d)$ with a metric that agrees with the definition of convergence in $\mathcal{S}(\mathbb{R}^d)$.

S2. $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

S3. For $a > 0$, $e^{-a\|x\|^2} \in \mathcal{S}(\mathbb{R}^d)$.

S4. If $f \in \mathcal{S}(\mathbb{R}^d)$ then $f \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

S5. Let $g \in \mathcal{S}(\mathbb{R}^d)$. The map $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ defined with $T(f) = gf$ is continuous.

S6. Let p be a polynomial. The map $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ defined with $T(f) = pf$ is continuous.

S7. Differentiation operator is continuous on $\mathcal{S}(\mathbb{R}^d)$.

Problems from the lecture

P1. Let H be a Hilbert space and $T : H \to H$ such that $\|Tx\| = \|x\|$. Prove that $\langle Tx, Ty\rangle = \langle x, y\rangle$.