Problem Set 10: convolutions and Schwartz sps

C1 $f \in L^1(\mathbb{R}^n)$, $g: \mathbb{R}^n \to \mathbb{R}$ Lipschitz i.e. $\underset{C_g}{\exists}$ $|g(x)-g(y)| \leq C_g|x-y|$.

$$\left| f*g(x) - f*g(y) \right| = \left| \int_{\mathbb{R}^n} f(z)\, g(x-z)\, dz - \int_{\mathbb{R}^n} f(z) g(y-z)\, dz \right| \leq$$

$$\leq \int_{\mathbb{R}^n} |f(z)| \underbrace{|g(x-z)-g(y-z)|}_{\leq C_g |x-y|} dz \leq C_g \|f\|_{L^1} |x-y|.$$

✓

C2 This is just revision. For all details, see

- P. Strzelecki "Mathematical Analysis II"
- L.C. Evans, M. Gariepy "Measure theory and fine properties of functions" Section 4.1.

this is somehow average of f with g determing weights

Let $f \in L^1$, $g \in C_0^k(\mathbb{R}^n)$. Note that $f*g(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy$

Roughly speaking, as x is only in g, differantiation does not see f. More precisely

$$f*g(x+h) - f*g(x) = \int_{\mathbb{R}^n} f(y) \left[\underbrace{g(x+h-y) - g(x-y)}_{\leq \|Dg\|_\infty \cdot |h|} \right] dy$$

So by Dominated Convergence Theorem ($f \in L^1$)

$$f*g(x+h) - f*g(x) - f*(Dg \cdot h)(x) \longrightarrow 0$$

So $D(f*g) = f*Dg$. Similarly higher derivatives ...

(C3) This is question about well-definiteness of convolution. In C4, which is part of Big Homework, you will prove directly Young's Inequality asserting that $\overbrace{\text{directly Young's}}^{\text{from Hölder}}$

$$f \in L^p, \quad g \in L^q \Rightarrow f * g \in L^r \qquad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

Theorem (Riesz, Thorin)

$$T: L^{p_0} \to L^{q_0} \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$$

$$T: L^{p_1} \to L^{q_1} \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0} \qquad \theta \in [0,1]$$

Then $T: L^{p_\theta} \to L^{q_\theta}$ and $\|T\|_{p_\theta, q_\theta} \leq \|T\|_{p_0,q_0}^{1-\theta} \|T\|_{p_1,q_1}^{\theta}$.

We prove two estimates:

(A) $\|f*g\|_\infty \leq \|f\|_p \|g\|_{p'}$

Indeed, $$\left| \int f(y)\, g(x-y)\, dy \right| \underset{\text{Hölder}}{\leq} \left(\int |f(y)|^p dy \right)^{1/p} \left(\int |g(x-y)|^{p'} dy \right)^{1/p'}$$

$$\underset{\uparrow \text{ change of var}}{=} \|f\|_p \|g\|_{p'}.$$

(B) $\|f*g\|_p \leq \|f\|_p \|g\|_1$

$$\left| \int \left| \int g(y) f(x-y)\, dy \right|^p dx \right|^{1/p} \underset{\uparrow \text{ Minkowski}}{\leq} \int \left| \int |g(y)|^p |f(x-y)|^p dx \right|^{1/p} dy =$$

$$= \int |g(y)| \left| \int |f(x-y)|^p dx \right|^{1/p} dy \leq \|g\|_1 \|f\|_p. \quad ✓$$

So if $f \in L^p$ is fixed $Tg = f * g$ is bounded as

$$T : L^{p'} \longrightarrow L^{\infty}$$
$$T : L^{1} \longrightarrow L^{p}$$

$\Rightarrow$ (fix $\theta \in [0,1]$)

$$\frac{1}{p_\theta} = \frac{\theta}{1} + \frac{1-\theta}{p'} = \theta + \frac{1-\theta}{p'}$$
$$\frac{1}{q_\theta} = \frac{\theta}{p} + \frac{1-\theta}{\infty} = \frac{\theta}{p}$$

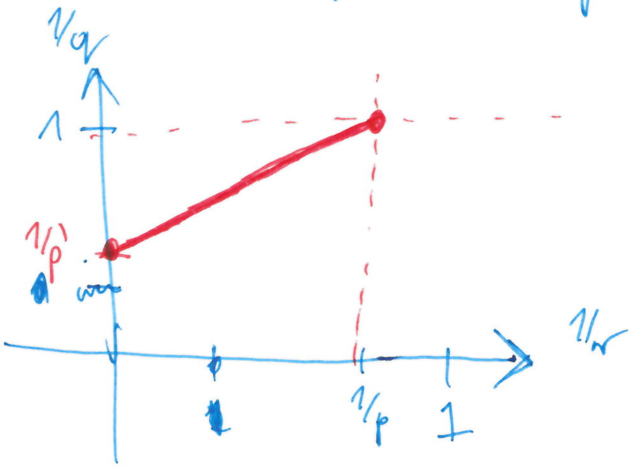
Note that $\frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow \frac{1}{p'} = 1 - \frac{1}{p} \Rightarrow \frac{1}{p_\theta} = \theta + (1-\theta) - \frac{1-\theta}{p} =$

$$= 1 + \frac{\theta}{p} - \frac{1}{p} = 1 + \frac{1}{q_\theta} - \frac{1}{p}$$

$\Rightarrow 1 + \frac{1}{q_\theta} = \frac{1}{p} + \frac{1}{p_\theta}$ and $\frac{\|f*g\|_{L^{q_\theta}}}{\|g\|_{p_\theta}} \leq \|f\|_p^{\theta} \|f\|_p^{1-\theta} = \|f\|_p$

$\Rightarrow \|f*g\|_{L^{q_\theta}} \leq \|f\|_p \|g\|_{p_\theta}$ where $1 + \frac{1}{q_\theta} = \frac{1}{p} + \frac{1}{p_\theta}$.

Note that as $\theta \in [0,1]$, $p_\theta \in [1, p']$, $q_\theta \in [p, \infty]$. Then, the remaining question is whether we have covered all exponents p, q, r s.t. $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Fix p and note that $\frac{1}{q} - \frac{1}{r} = \frac{1}{p'}$ so for fixed p q changes from p' (for $r = \infty$) to 1 (for $r = p$). Parameter r cannot get smaller then p as then q would be smaller than 1. This completes the proof.



Remark:
This is geometric way of doing R-T theorem. We know that $(0, \frac{1}{p'})$ and $(\frac{1}{p}, 1)$ are admissible pairs so we conclude that the whole line is also admissible.

C4 (Big Homework 4)

C5 $\eta \in C_0^\infty(B_1(0))$, $\int \eta = 1$, $\eta \geq 0$, $\eta_\varepsilon = \varepsilon^{-d}\eta\left(\frac{x}{\varepsilon}\right)$

so $\int \eta_\varepsilon = 1$.

We take $\tilde{f} = \mathbb{1}_{B_{3/2}(0)}$. Then, we mollify it with $\eta_{1/8}$.

$$\tilde{f} * \eta_\varepsilon = \int \tilde{f}(y)\,\eta_\varepsilon(x-y)\,dy = \int_{B(0,\frac{1}{8})} \tilde{f}(x-y)\,\eta_\varepsilon(y)\,dy$$

$= 1$ for $x \in B(0,1)$.

Similarly $\tilde{f} * \eta_\varepsilon = 0$ for $x \in B(0,2)$.

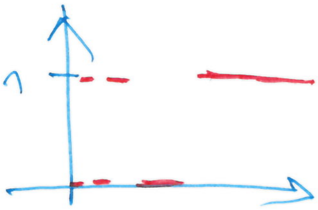
Finally $|\tilde{f} * \eta_\varepsilon| \leq \|\tilde{f}\|_\infty \int \eta_\varepsilon = \|\tilde{f}\|_\infty = 1$. so

$\tilde{f} * \eta_\varepsilon \in [0,1]$.

$\square$.

C6 Consider a function oscillating very quickly (from $L^\infty(0,1)$).

$$u(x) = \begin{cases} 1 & x \in \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right] \quad n \in \mathbb{N} \\ 0 & x \in \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}}\right) \end{cases}$$



Let $f \in C[0,1]$ s.t. $\|f-u\|_\infty \leq \frac{1}{4}$ i.e. $\forall_{x \in [0,1]}\ |f(x)-u(x)| \leq \frac{1}{4}$.

Using continuity of f and definition of u,

$|f(0)-1| \leq \frac{1}{4}$, $|f(0)| \leq \frac{1}{4}$. $\Rightarrow$ contradiction.

S1 $\quad p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)|$

Let $f(\alpha,\beta): \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{N}$ be bijection. We claim that

$$d(f,g) = \sum_{\alpha,\beta} 2^{-f(\alpha,\beta)} \left[p_{\alpha,\beta}(f-g) \Big/ 1 + p_{\alpha,\beta}(f-g) \right]$$

metriceses convergence in $S(\mathbb{R}^d)$.

- $d(f_n, f) \to 0 \Rightarrow p_{\alpha,\beta}(f-f_n) \to 0 \quad \forall_{\alpha,\beta}$. ✓
- $\forall_{\alpha,\beta}\ p_{\alpha,\beta}(f-f_n) \to 0$. We note that

$$\left| \frac{p_{\alpha,\beta}(f-f_n)}{1+p_{\alpha,\beta}(f-f_n)} \right| \le \min(1, p_{\alpha,\beta}(f-f_n))$$

Fix $\varepsilon > 0$.
We find a finite set of (α,β) den. with A_ε s.t.

$\sum_{\alpha,\beta} 2^{-f(\alpha,\beta)} \le \varepsilon/2$. For each $(\alpha,\beta)\in A_\varepsilon$ we find $N_{\alpha,\beta}$ s.t.

$\forall_{n\ge N_{\alpha,\beta}}\ p_{\alpha,\beta}(f-f_n) \le \frac{\varepsilon}{2}$. Then, $\forall_{n \ge \max_{\alpha,\beta\in A_\varepsilon} N_{\alpha,\beta}}$

$$d(f,f_n) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

✓

(S2) If $f \in C_0^\infty(\Omega)$ then f has compact support and function with compact support has finite supremum. That's why $C_0^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$

(S3) For $a>0$, $e^{-a\|x\|^2} \in S(\mathbb{R}^d)$.

It is sufficient to prove that $\sup_{x \in \mathbb{R}^d} \left| p(x) e^{-a\|x\|^2} \right|$ is finite for any polynomial.

$$p(x) e^{-a\|x\|^2} = \frac{p(x)}{e^{a\|x\|^2}}$$

(By Taylor, $e^{a\|x\|^2} \geq 1 + C_N \|x\|^N$ for any $N \in \mathbb{N}$). ✓

(S4) $f \in S(\mathbb{R}^d) \Rightarrow f \in L^p(\mathbb{R}^d)$ $\forall_{1 \leq p \leq \infty}$

$p = \infty$ done as f is bounded.

$$\int_{\mathbb{R}^d} |f|^p \leq \underbrace{\int_{B(0,1)} |f|^p}_{\leq \|f\|_\infty^p \, |B(0,1)|} + \int_{\mathbb{R}^d \setminus B(0,1)} |f|^p$$

on this part we use that $|f| \, |x|^\alpha \leq C_\alpha$ for any α.

$$\int_{\mathbb{R}^d \setminus B(0,1)} |f|^p \leq \int_{\mathbb{R}^d \setminus B(0,1)} \frac{C_\alpha^p}{|x|^{p\alpha}} = C_\alpha^p \int_1^\infty \frac{1}{r^{p\alpha}} C \cdot r^{d-1} \, dr = \tilde{C} \int_1^\infty r^{d-1-p\alpha} \, dr$$

We want $d-1-p\alpha < -1 \Rightarrow \alpha > \frac{d}{p}$ will work to make this integral finite.

S5 We let $f_n \to f$, $Tf = gf$ for some $g \in S(\mathbb{R}^d)$ and we have to check $Tf_n \to Tf$ in $S(\mathbb{R}^d)$ i.e. $\forall_{\alpha,\beta}$

$p_{\alpha,\beta}(Tf_n - Tf) \to 0$. $\to$ $\sup_x |x^\alpha| D^\beta [f_n g - fg]||$

$= \sup_x |x^\alpha D^\beta (f_n - f) g|$ ...

S6 Similar as S5 with polynomial this time...

S7 Similar as S5.

P1 $\|Tx\| = \|x\|$ for all x

Note that $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle^2 + \langle y,y\rangle^2 +$

$+ \langle x,y\rangle + \overline{\langle x,y\rangle}$

$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \overline{\langle x,y\rangle}$

$\Rightarrow 4\,\mathrm{Re}\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$ so real part is preserved

Then, consider $x + iy$ to deduce that $\mathrm{Im}\langle x,y\rangle$ is also preserved.