Functional Analysis (WS 19/20), Problem Set 11

(Fourier series, Fourier transform and tempered distributions*)

If $f \in L^1(0,1)$, we define **Fourier series** of f with its partial sums

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x}, \qquad \hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\, dx.$$

It is a convention to extend periodically f to the whole $\mathbb{R}$.

For $f \in L^1(\mathbb{R}^n)$ we define **Fourier transform** of f with

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}.$$

A **tempered distribution** is a continuous linear functional on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The space of all tempered distributions is denoted with $\mathcal{S}'(\mathbb{R}^n)$. If $T \in \mathcal{S}'(\mathbb{R}^n)$, we define Fourier transform of T as $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\hat{T}(f) = T(\hat{f}) \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

This definition makes sense as it is well-known that the Fourier transform is isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Fourier series

The target of the exercises below is to formulate some conditions on f so that $S_N f(x) \to f(x)$ in appropriate sense.

S1. Suppose you are a student at the Faculty of Physics. Derive Fourier series of f, i.e. find decomposition of f into sin and cos functions.

S2. Prove that $S_N f(x) = \int_0^1 f(x-t) D_N(t)$ where D_N is the Dirichlet kernel defined with

$$D_N(x) = \sum_{k=-N}^{k=N} e^{2\pi i k x}.$$

S3. Prove the following properties of Dirichlet kernel:

(A) $D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}$,

(B) $\int_0^1 D_N(t)\, dt = 1$,

(C) for t such that $\delta \le |t| \le \frac{1}{2}$, $|D_N(t)| \le \frac{1}{\sin \pi\delta}$.

S4. (**Riemman-Lebesgue Lemma**) We have $|\hat{f}(k)| \le \|f\|_{L^1(0,1)}$ and even better, we have $\lim_{k\to\pm\infty} |\hat{f}(k)| = 0$.

S5. (**Riemman Localization Principle**) Let $f \in L^1(0,1)$. If $f = 0$ in some neighbourhood of x, then $S_N f(x) \to 0$.

S6. (**Dini Theorem**) Fix $x \in \mathbb{R}$. Let f be a continuous function that is 1-periodic, i.e. $f(y+1) = f(y)$ for all $y \in \mathbb{R}$. Suppose that f satisfies for some $\delta > 0$ the following condition:

$$\int_{|t|<\delta} \frac{|f(x+t) - f(x)|}{|t|}\, dt < \infty.$$

Use properties of the Dirichlet kernel and the proof of Riemann Localization Principle to prove that the Fourier series of f converges at x to $f(x)$.

S7. Continuity of f is not sufficient. There is a continuous function on $[0,1]$ such that $S_N f(0)$ diverges to ∞, see Special Problem 10.

S8. Prove that if $f \in L^2(0,1)$ then $S_N f \to f$ in $L^2(0,1)$.

Fourier transform for $f \in L^1, f \in L^2, f \in \mathcal{S}$

T1. Fourier transform is linear.

T2. We have $\|\hat{f}\| \le \|f\|_1$ and $\hat{f}$ is continuous.

T3. ☺(**Riemman-Lebesgue Lemma**) We have $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

T4. ☺Convolution becomes multiplication: $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

T5. ☺Translation becomes rotation: $\widehat{\tau_h f}(\xi) = \hat{f}(\xi)e^{2\pi i \xi \cdot h}$ where $\tau_h f(x) = f(x+h)$.

T6. ☺Differentiation becomes multiplication by a polynomial: $\widehat{f_{x_j}}(\xi) = 2\pi i \xi \hat{f}(\xi)$.

T7. ☺Let $\hat{f}$ be the Fourier transform of $f \in L^1(\mathbb{R}^n)$. Find $\widehat{\delta_h f}$ where $\delta_h f(x) = f(x/h)$.

T8. ☺Compute $\hat{f}$ for $f(x) = e^{-\pi|x|^2}$.

T9. ☺Compute $\hat{f}$ (in one dimension) for $f(x) = e^{-x}\chi_{[0,\infty)}(x)$. This is important!

T10. (**Plancherel theorem**) If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f(x)}\,\overline{\widehat{g(x)}}\, dx.$$

T11. Fourier transform is a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. The inverse is given with

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i \xi \cdot x}\, d\xi.$$

T12. Fourier transform extends on $L^2(\mathbb{R}^n)$ by density argument and is an isometrical isomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

T13. ☺Compare Fourier transform on $L^1(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ in view of definition, image and possibility to invert the transform.

T14. ☺Let $f \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Solve the PDE $-\Delta u - u = f$ in $\mathbb{R}^n$.

T15. ☺(**Heisenberg uncertainity principle**) Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\|\psi\|_2 = 1$. Prove that

$$\left[\int_{\mathbb{R}} x^2|\psi(x)|\, dx\right] \cdot \left[\int_{\mathbb{R}} \xi^2|\hat{\psi}(\xi)|^2\, d\xi\right] \ge \frac{1}{16\pi^2}.$$

T16. ☺Let $g \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$.

(A) Let $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be given with $Mf = \hat{g}f$. Prove that M is well-defined, i.e. it has image in $L^2(\mathbb{R})$.

(B) Prove that $\sigma(M) = \overline{\{\hat{g}(x) : x \in \mathbb{R}\}} = \{\hat{g}(x) : x \in \mathbb{R}\}$.

(C) Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined with $Tf = f * g$. Prove that T is well-defined.

(D) Find $\sigma(T)$.

T17. (**Hausdorff-Young Lemma**) Use Riesz-Thorin interpolation theorem to deduce that the Fourier transform extends to a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ where $p \in (1, 2)$.

Tempered distributions

TD1. Prove that Fourier transform is a continuous isomorphism between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

TD2. Prove that if $f \in L^p(\mathbb{R}^n)$ for any $p \in [1, \infty]$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ under the canonical embedding

$$L^p(\mathbb{R}^n) \ni f \mapsto I_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathcal{S}'(\mathbb{R}^n).$$

TD3. Let $f \in L^1(\mathbb{R}^n)$. In what sense the following two objects coincide:

(A) Fourier transform of f computed directly from the definition for $L^1(\mathbb{R}^n)$ functions,

(B) Fourier transform of f computed for f treated as a tempered distribution.

TD4. In view of TD2., one can compute Fourier transform for any L^p function rather than just for L^q function for $q \in [1, 2]$. What is the price for that?