Problem Set 11

(Fourier Series, Fourier transform, tempered dist.)

(S1) Let f be periodic on $\mathbb{R}$ with period 1. As $\{e^{2\pi i kx}\}$ is basis of $L^2(0,1)$ we use decomposition

$$f(x)=\sum_{k=-\infty}^{\infty} c_k \cdot e^{2\pi i kx}, \quad c_k = \langle f, e^{2\pi i kx}\rangle$$

We have $c_k = \langle f, e^{2\pi i kx}\rangle = \int_0^1 f(x)\overline{e^{2\pi i kx}}\,dx =$

$$\overline{e^{2\pi i kx}} = \cos(2\pi kx) - i\sin(2\pi kx)$$

$$= \int_0^1 f(x) e^{-2\pi i kx}\,dx.$$

We use notation $\hat{f}(k) := \int_0^1 f(x)e^{-2\pi i kx}\,dx$ so in some sense we have $f(x)=\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{2\pi i kx}$. By Hilbert space theory, this series converges in $L^2(0,1)$ for any $f\in L^2(0,1)$ [this is consequence of a standard fact that $\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ converges in H to x where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal Schauder basis of H].

Historical notes on this problem: It was a long-standing problem in which other sense (not L^2) Fourier series of f converges. Here is some review:

- $f\in L^2(0,1) \Rightarrow S_N f \to f$ in $L^2(0,1)$. [That's what we have done above]

→ If f is just continuous, $S_N f$ can diverge [explicit construction due to du Bois-Reymond (1873)] but see also Special Problem 10 for topological proof.

→ If f is sufficiently uniformly continuous, $S_N f(x) \to f(x)$ pointwise (Dini, Jordan). This is what we're gonna prove.

→ If $f \in L^1(0,1)$ $S_N f \to f$ does not hold in L^1 (Kolmogorov, 1926) or even a.e.

→ If $f \in L^p(0,1)$, $p \in (1,\infty)$, $S_N f \to f$ converges pointwise (a.e.) and in $L^p(0,1)$. ((Carleson, Hunt, '70s).

S2

$$S_N f = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x} = \sum_{k=-N}^{N} \left(\int_0^1 f(t) e^{-2\pi i k t} dt \right) e^{2\pi i k x}$$

$$= \sum_{k=-N}^{N} \int_0^1 f(t) e^{-2\pi i k (t-x)} dt = \int_0^1 f(t) D_N(x-t)\, dt$$

period: 1.

$$= \int_{-x}^{1-x} f(x-t) D_N(t)\, dt \underset{\text{period } 1}{=} \int_0^1 f(x-t) D_N(t)\, dt.$$

S3 (A) $D_N(t) = \dfrac{\sin(\pi(2N+1)t)}{\sin(\pi t)}$. Indeed, $D_N(x) = \sum_{k=-N}^{N} e^{2\pi i k x} =$

$$= e^{-2\pi i N x} \sum_{k=-N}^{N} e^{2\pi i (k+N) x} = \left(\sum_{k=0}^{2N} e^{2\pi i k x} \right) e^{-2\pi i N x} =$$

$$= \frac{1 - e^{2\pi i x(2N+1)}}{1 - e^{2\pi i x}} \cdot e^{-2\pi i N x} = \frac{\overbrace{e^{\pi i x(2N+1)} - e^{-\pi i x(2N+1)}}^{= 2i \sin(\pi(2N+1)x)}}{\underbrace{e^{\pi i x} - e^{-\pi i x}}_{= 2i \sin(\pi x)}} \underbrace{e^{-2\pi i N x} e^{\pi i x} e^{-\pi i x(2N+1)}}_{= 1}.$$

(B) $\int_0^1 D_N(t)\,dt = 1$ as $\int_0^1 e^{2\pi i k x}\,dx = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$.

(C) As $D_N(t) = \dfrac{\sin((2N+1)\pi t)}{\sin(\pi t)}$, let $t \in (\delta, \frac{1}{2})$ then

$$|D_N(t)| \leqslant \frac{1}{|\sin(\pi t)|} \leqslant \frac{1}{\sin(\pi\delta)}.$$

S4 Clearly $|\hat{f}(k)| = \left| \int_0^1 f(x) e^{-2\pi i k x} \right| \leqslant \|f\|_1$.

Another statement is more tricky. We have $\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x}\,dx$.

Note that $e^{-\pi i} = -1$ so we also have

$$\hat{f}(k) = -\int_0^1 f(x) e^{-2\pi i k x} e^{-\pi i} = -\int_0^1 f(x) e^{-2\pi i k \left[x - \frac{1}{2k}\right]}\,dx$$

$$= -\int_0^1 f\left(t - \frac{1}{2k}\right) e^{-2\pi i k t}\,dt$$

$\Rightarrow \hat{f}(k) = \frac{1}{2}\int_0^1 \left[f(t) - f\left(t - \frac{1}{2k}\right)\right] e^{-2\pi i k t} \longrightarrow 0$ for f continuous.

Let $f \in L^1(0,1)$. There is a sequence $f_n \in C(0,1)$, $f_n \to f$ in L^1.

$$\hat{f}(k) = \int_0^1 f(t) e^{-2\pi i k t}\,dt \leq \int_0^1 |f(t) - f_n(t)|\, e^{-2\pi i k t}\,dt + |\hat{f}_n(k)|$$

$$\leqslant \|f - f_n\|_1 + |\hat{f}_n(k)|.$$

$\Rightarrow \limsup_{k\to\infty} |\hat{f}(k)| \leqslant \|f - f_n\|_1 \leftarrow$ arbitrarily small

$\Rightarrow \limsup_{k\to\infty} |\hat{f}(k)| = 0 \Rightarrow \hat{f}(k) \longrightarrow 0$ as $k \to \infty$.

$\square$.

S5 Suppose that $f(t)=0$ on $(x-\delta, x+\delta)$. Note that

$$S_N f(x) = \int_{-1/2}^{1/2} f(x-t)\,\frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\,dt =$$

$$= \int_{\delta<|t|\leq 1/2} f(x-t)\,\frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\,dt = (*)$$

We have $\sin(\pi(2N+1)t) = \frac{1}{2i}\left[i\sin(\ldots)+\cos(\ldots)-\underset{\cos(\ldots)}{\underset{\|}{\cos(-\ldots)}}-i\sin(-\ldots)\right]$

$$= \frac{1}{2i}\left[e^{\ldots}-e^{-\ldots}\right]$$

$$(*) = \int_{\delta<|t|<\frac{1}{2}} f(x-t)/\sin\pi t\,(2i)\; e^{2\pi i N t}\cdot e^{\pi i t}\,dt -$$

$$- \int_{\delta<|t|<\frac{1}{2}} f(x-t)/\sin(\pi t)(2i)\; e^{-2\pi i N t}\, e^{-\pi i t}\,dt. = (*)$$

Let $g_x(t) = \mathbb{1}_{\{\delta<|t|<\frac{1}{2}\}}\,\frac{f(x-t)}{2i\sin(\pi t)}$. Then

$$(*) = \left(\widehat{g_x e^{\pi i\cdot}}\right)(N) - \left(\widehat{g_x e^{\pi i\cdot}}\right)(-N) \longrightarrow 0 \text{ as } N\to\infty.$$

by R-L Lemma

S6 Part of Big Homework 5.

S7 Special Problem 10.

S8 We solved this in S1.

Literature:
- basics: J. Duoandikoetxea "Fourier Analysis" chap. 1
- more adv. topics Grafakos "Classical Fourier Analysis" chap. 2.2–2.4

T1 Clearly, Fourier transform is linear.

T2 $\left| \int_{\mathbb{R}^n} f(x) \underbrace{e^{-2\pi i \xi \cdot x}}_{|\dots| \leq 1} dx \right| \leq \|f\|_1$

Let $\xi_n \to \xi$ in $\mathbb{R}^n$. We want $\hat{f}(\xi_n) \to \hat{f}(\xi)$. Indeed,

$$\hat{f}(\xi_n) = \int_{\mathbb{R}^n} \underbrace{f(x) e^{-2\pi i \xi_n \cdot x}}_{\substack{\text{converges pointwise to} \\ f(x)e^{-2\pi i \xi x}}} dx \longrightarrow \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = \hat{f}(\xi)$$

by DCT.

; integrable majorant is f

T3 $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ $\quad (f \in L^1(\mathbb{R}^n))$.

This is proved with density argument. Clearly, ~~$C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Moreover, functions of the form $\sum_{i=1}^{n} a_i \dots$ simple fcts are dense in $L^1(\mathbb{R}^n)$. Moreover, bst measurable subsets in the fixed simple fcn, we can take cubes as they generate $\mathcal{L}(\mathbb{R}^n)$. So it is sufficient to prove the assertion for $f = 1_Q$, $Q = (a_1, b_1) \times \dots \times (a_n, b_n)$. However~~

$$\overline{C_c^\infty(\mathbb{R}^n)}^{L^1} = L^1(\mathbb{R}^n) \quad \left(\text{i.e. } C_c^\infty(\mathbb{R}^n) \text{ is dense in } L^1(\mathbb{R}^n)\right).$$

Let $\xi \in \mathbb{R}^n$, $|\xi| \to \infty$. In particular, $\exists_{i \in \{1, \dots, n\}}$ $|\xi_i| \to \infty$.

Then, if $f \in C_c^\infty(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n} f(x) \frac{1}{(-2\pi i \xi_i)} \partial_{x_i}\left[e^{-2\pi i \xi x}\right] dx$$

$$\underset{\substack{\uparrow \\ \text{integration by} \\ \text{parts}}}{=} -\frac{1}{2\pi i \xi_i} \int_{\mathbb{R}^n} \partial_{x_i} f(x)\, e^{-2\pi i \xi x} dx \leq \frac{1}{2\pi |\xi_i|} \underbrace{\|\partial_{x_i} f\|_{L^1}}_{\text{finite as } f \in C_c^\infty(\mathbb{R}^n)} \longrightarrow 0$$

The general statement follows by density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ (this time check it yourself!).

Ok: Let $f_n \in C_c^\infty(\mathbb{R}^n)$, $f_n \to f$ in $L^1(\mathbb{R}^n)$, $\|\hat{f}_n - \hat{f}\|_1 \to 0, \dots$.

Integration by parts (Analysis II): The following is not stressed during Analysis course. By Stokes theorem, if $F : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, Ω has suff. smooth boundary we have

$$\int_\Omega \underset{\substack{\uparrow \\ F_{x_1} + \dots + F_{x_n}.}}{\operatorname{div} F}\, dx = \int_{\partial\Omega} F \cdot \underset{\substack{\uparrow \\ \text{outward normal vector}}}{n}\, dS$$

If $g : \mathbb{R}^n \to \mathbb{R}$ then $\operatorname{div}(Fg) = (\operatorname{div} F) g + F \cdot \nabla g$. We get:

$$\boxed{\int_\Omega (\operatorname{div} F) g + \int_\Omega F \cdot \nabla g = \int_{\partial\Omega} Fg \cdot n\, dS}$$

Finally, if $F = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{f}, 0, \dots, 0)$, where $f : \mathbb{R}^n \to \mathbb{R}$ we obtain

$$\boxed{\int_\Omega f_{x_i} \cdot g + \int_\Omega f \cdot g_{x_i} = \int_{\partial\Omega} fg \cdot n_i\, dS}$$

This is integration by parts in $\mathbb{R}^n$. To extend it to bounded domains, suppose that $f_{x_i} \cdot g \in L^1$, $f \cdot g_{x_i} \in L^1$, and one of

f or g has compact support. By $\Omega = B(0,r)$ and sending $r \to \infty$ we get:

$$\boxed{\int_{\mathbb{R}^n} f_{x_i} g \, dx = - \int_{\mathbb{R}^n} f g_{x_i} \, dx}$$

!!!

T4 $f * g(x) \in L^1(\mathbb{R}^n)$ if $f, g \in L^1(\mathbb{R}^n)$ [Young's inequality]

$$\widehat{f*g}(\xi) = \int_{\mathbb{R}^n} f*g(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) g(x-y) e^{-2\pi i \xi x} dy \, dx$$

$$= \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} g(x-y) e^{-2\pi i \xi (x-y)} dx \right] e^{-2\pi i \xi y} dy =$$

$$= \left[\int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi y} dy \right] \hat{g}(\xi) = \hat{f}(\xi) \hat{g}(\xi).$$

So $\boxed{f, g \in L^1(\mathbb{R}^n)} \Rightarrow f*g \in L^1(\mathbb{R}^n)$ and $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

T5 $\tau_h f(x) = f(x+h)$ $\boxed{f \in L^1(\mathbb{R}^n)}$

$$\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^n} \tau_h f(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n} f(x+h) e^{-2\pi i \xi x} dx =$$

$$= \left[\int_{\mathbb{R}^n} f(x+h) e^{-2\pi i \xi (x+h)} dx \right] e^{2\pi i \xi h} = \hat{f}(\xi) e^{2\pi i \xi h}.$$

(T6) $f \in L^1(\mathbb{R}^n)$, $\partial_{x_j} f \in L^1(\mathbb{R}^n)$, f vanishes at ∞ suff fast ($f \in \mathcal{S}(\mathbb{R}^n)$ works)

$$\widehat{\partial_{x_j} f}(\xi) = \int_{\mathbb{R}^n} \partial_{x_j} f(x) e^{-2\pi i \xi x} = -\int_{\mathbb{R}^n} f(x)(-2\pi i \xi_j) e^{-2\pi i \xi x} dx =$$

$$= 2\pi i \xi_j \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = 2\pi i \xi_j \hat{f}(\xi)$$

Remark: One quickly observes that the computations above require hard-to-remember assumptions. In principle, one usually starts with proving all of them for $f \in \mathcal{S}(\mathbb{R}^n)$ (no technical difficulties) and then moving to the space of tempered distributions.

for this, see Grafakos "Classical Fourier Anal" - Chapt. 2.2-2.4

(T7) $\delta_h f = f(x/h) \quad f \in L^1(\mathbb{R}^n)$

$$\widehat{\delta_h f}(\xi) = \int_{\mathbb{R}^n} \delta_h f(x) e^{-2\pi i \xi x} dx =$$

$$= \int_{\mathbb{R}^n} f(x/h) e^{-2\pi i \xi x} dx \underset{y = \frac{x}{h}}{=} \left[\int_{\mathbb{R}^n} f(y) e^{-2\pi i (\xi h) \cdot y} dy \right] h^n$$

$$= h^n \hat{f}(\xi h).$$

(T8) We first note that one-dimensional case is sufficient. Indeed, $f = e^{-\pi |x|^2}$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\pi(x_1^2 + \dots + x_n^2)} e^{-2\pi i(\xi_1 x_1 + \dots + \xi_n x_n)} =$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\pi x_i^2} e^{-2\pi i \xi_i x_i} dx_i$$

So we work on $\mathbb{R}$ instead of $\mathbb{R}^n$, $f(x)=e^{-\pi x^2}$, $f:\mathbb{R}\to\mathbb{R}$.

f satisfies $\begin{cases} f' + 2\pi x f(x) = 0 \\ f(0)=1. \end{cases}$ $(*)$

Moreover, $\hat{f}(0) = \int_{\mathbb{R}^n} e^{-\pi x^2} = 1$. We want to prove that $\hat{f}$ also solves $(*)$. Indeed,

- $\frac{d}{d\xi}\hat{f}(\xi) = \int_{\mathbb{R}} f(x)(-2\pi i x)e^{-2\pi i \xi x}\,dx = \widehat{f(-2\pi i x)}(\xi)$.
- $2\pi\xi\hat{f}(\xi) = +\frac{1}{i}(2\pi\xi i)\hat{f}(\xi) = +\frac{1}{i}\hat{f_x}(\xi) = -i\hat{f_x}(\xi)$.

Therefore $\frac{d}{d\xi}\hat{f}(\xi) + 2\pi\xi\hat{f}(\xi) = \left(-2\pi i x f(x) - i f_x\right)^{\wedge}(\xi) = 0$

By uniqueness of slns to $(*)$, the assertion follows. $\square$

T9 $f(x) = e^{-x}\mathbb{1}_{x>0}$ $f:\mathbb{R}\to\mathbb{R}$

$$\hat{f}(\xi) = \int_0^\infty e^{-x}e^{-2\pi i\xi x}\,dx = \int_0^\infty e^{-x(2\pi i\xi+1)}\,dx =$$

$$= \frac{1}{1+2\pi i\xi} \notin L^1(\mathbb{R})$$

So it is not true that $f\in L^1(\mathbb{R}^n) \Rightarrow \hat{f}\in L^1(\mathbb{R}^n)$.

T10, T11, T12 (lecture)

B $f,g\in S(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} f\bar{g} = \int_{\mathbb{R}^n}\hat{f}\bar{\hat{g}}$. In particular, if $f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

Ⓐ Fourier transform can be inverted on $S(\mathbb{R}^n)$ with

$$\check{f}(\cancel{x}) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi x} dx \longrightarrow$$ this formula is well-def for f in $L^1(\mathbb{R}^n)$.

This shows that Fourier transform is not invertible with that formula due to Problem T9 $(\hat{f} \notin L^1,\ f \in L^1)$. on $L^1(\mathbb{R}^n)$

Ⓒ **Definition of Fourier transform on L^2:** If $f \in L^2(\mathbb{R}^n)$, then

$f \mathbb{1}_{\{|\cancel{x}| \le \cancel{k}\}} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

Let $f_r = f \mathbb{1}_{\{|\cancel{x}| \le r\}}$.

We know that $f_r \longrightarrow f$ in L^2 by DCT. Moreover, by Plancherel,

$\|f_r\|_{L^2} = \|\hat{f}_r\|_{L^2}$ and as L^2 is a Banach space, $\{\hat{f}_r\}$ has a limit in $L^2(\mathbb{R}^n)$. We define

$$\hat{f}(\xi) = \lim_{r \to \infty} \int_{B(0,r)} f(x) e^{-2\pi i \xi x} dx, \quad (*)$$

the limit is taken in $L^2(\mathbb{R}^n)$ sense

It follows that

$$\check{f}(x) = \lim_{r \to \infty} \int_{B(0,r)} f(\xi) e^{+2\pi i \xi x} d\xi,$$

the limit is taken again in $L^2(\mathbb{R}^n)$ sense

Remark: one can take any $\{f_n\} \in L^2 \cap L^1$ s.t. $f_n \to f$ in L^2 to define $\hat{f}$.

Ⓓ Properties of Fourier transform on $L^2(\mathbb{R}^n)$.

Most of them follows by def. $(*)$. For instance convolutions, see Problem T16 and convolution property.

T13

	definition	image	invertibility
$L^1(\mathbb{R}^n)$	$\hat{f}(\xi)=\int f(x)e^{-2\pi i \xi x}\,dx$	$C_0(\mathbb{R}^n)$ but not necessarily $L^1(\mathbb{R}^n)$	NO: see image and T9
$S(\mathbb{R}^n)$	—— $\|$ —— as $S(\mathbb{R}^n)\subset L^1(\mathbb{R}^n)$	$S(\mathbb{R}^n)$ (isomorphism)	YES, $\check{f}(x)=\int f(\xi)e^{2\pi i \xi x}$
$L^2(\mathbb{R}^n)$	density: $\hat{f}=\lim \widehat{f_r}$ where $f_r=f\,\mathbb{1}_{\{\lvert x\rvert\le r\}}$	$L^2(\mathbb{R}^n)$ (isomorphism isometry)	YES $\check{f}(x)=\lim f_r(\xi)$ limit in L^2 sense

T14 Let $f\in S(\mathbb{R}^n)$. We find $u\in S(\mathbb{R}^n)$ s.t.

$$-\Delta u+u=f$$

First, take Fourier transform to get $\hat{f}(\xi)=4\pi^2|\xi|^2\hat{u}(\xi)+\hat{u}(\xi)$

$$\Rightarrow \hat{u}(\xi)=\frac{1}{4\pi^2|\xi|^2+1}\hat{f}(\xi).$$

As $f\in S(\mathbb{R}^n)$, $\Rightarrow \hat{f}(\xi)\in S(\mathbb{R}^n) \Rightarrow \dfrac{\hat{f}(\xi)}{1+4\pi^2|\xi|^2}\in S(\mathbb{R}^n)$.

Since Fourier transform is isomorphism on $S(\mathbb{R}^n)$, there is $u\in S(\mathbb{R}^n)$ s.t. $\hat{u}(\xi)=\dfrac{\hat{f}(\xi)}{1+4\pi^2|\xi|^2}$. We write

$$u(x)=\left(\frac{\hat{f}(\xi)}{1+4\pi^2|\xi|^2}\right)^{\vee}.$$

T15 $1=\int 1\cdot|\psi|^2=-\int x\frac{d}{dx}|\psi|^2=-\int x\frac{d}{dx}\varphi\overline{\varphi}\,dx$

$$=-\int x\,\varphi_x\overline{\varphi}-\int x\,\varphi\,\overline{\varphi}_x\le 2\left(\int|x\,\varphi(x)|^2\right)^{1/2}\left(\int|\varphi_x|^2\right)^{1/2}$$

~~But: $\varphi_x(x)\neq\frac{d}{dx}\varphi$~~

By Plancherel $\int|\varphi_x|^2=\int|\widehat{\varphi_x}|^2=4\pi^2\int|\hat{\varphi}(\xi)|\xi||^2$

so that $1 \leqslant 2 \cdot (2\pi) \left(\int |x \psi(x)|^2 \right)^{1/2} \left(\int |\xi \hat{\psi}(\xi)|^2 \right)^{1/2}$

$$\Rightarrow \quad \frac{1}{16\pi^2} \leqslant \left(\int |x\psi(x)|^2 \right)^{1/2} \left(\int |\xi \hat{\psi}(\xi)|^2 \right)^{1/2} \qquad \square.$$

T16 $g \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

(A) $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. $Mf = \hat{g} f$. As $\hat{g} \in C_0(\mathbb{R}^n)$ (here $g \in L^1(\mathbb{R}^n)$ is sufficient) we obtain that $\hat{g} \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.
Therefore $\|Mf\|_{L^2} \leqslant \|\hat{g}\|_{L^\infty} \|f\|_{L^2} \leqslant \|g\|_{L^1} \|f\|_{L^2}$ so
$\|M\| \leqslant \|g\|_{L^1} \Rightarrow M$ is well-def. bdd operator.

(B) We know that $\sigma(M) = \overline{\{\hat{g}(x) : x \in \mathbb{R}\}}$. (see Problem S12).
~~As $\hat{g} \in C_0(\mathbb{R}^n)$ this is the same as $\{\hat{g}(x) : x \in \mathbb{R}^n\}$.~~

(C) $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ $\quad Tf = f * g \quad (g \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n))$.
Here, again $g \in L^1(\mathbb{R}^n)$ is sufficient. Indeed, by Young's ineq.
$1 + \frac{1}{2} = \frac{1}{2} + 1$ so $f \in L^2, g \in L^1 \Rightarrow f * g \in L^2$.

(D) We want to find $\sigma(T)$. We study invertibility of the operator $T - \lambda I$, $\lambda \in \mathbb{C}$. Let $\mathcal{F} : L^2 \to L^2$ be Fourier transform. As it is isomorphism, invertibility of $T - \lambda I$ is equivalent with invertibility of $\mathcal{F} \circ (T - \lambda I) \mathcal{F}^{-1}$ which is

$$\mathcal{F} \circ (T - \lambda I) \mathcal{F}^{-1} f = \mathcal{F} \circ (\check{f} * g) = f \hat{g}$$ so we have to find

$\sigma(f \mapsto f\hat{g})$ and this is given by (B).

Warning: We used here equality

$$\mathcal{F}(f * g) = f\,\hat{g} \quad \longrightarrow \text{ this is true for } f, g \in S(\mathbb{R}^n)$$
or at least if $f \in L^1(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$.

Question: if $f \in L^2(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$ so that $f * g \in L^2(\mathbb{R}^n)$

do we have $\widehat{f * g} = \hat{f} \cdot \hat{g}$? Indeed, we do.

Let $f_n \in C_c^\infty(\mathbb{R}^n)$, $f_n \to f$ in $L^2(\mathbb{R}^n)$. We have $f_n \in L^1(\mathbb{R}^n)$ and so $\widehat{f_n * g} = \hat{f_n}\,\hat{g}$.

Clearly, by Plancherel $\|\hat{f_n} - \hat{f}\|_{L^2} = \|f_n - f\|_{L^2} \longrightarrow 0$. As $\hat{g} \in L^\infty(\mathbb{R}^n)$, $\|\hat{g}(\hat{f_n} - \hat{f})\|_{L^2} \leq \|\hat{g}\|_\infty \|\hat{f_n} - \hat{f}\|_{L^2} \longrightarrow 0$ so that $\hat{g}\hat{f_n} \to \hat{g}\hat{f}$ in L^2. Similarly, by Young's inequality

$$\|f_n * g - f * g\|_2 \leq \|g\|_1 \|f_n - f\|_{L^2} \longrightarrow 0 \quad \text{so } f_n * g \to f * g \text{ in}$$

L^2. Again, by Plancherel $\|\widehat{f_n * g} - \widehat{f * g}\|_{L^2} \longrightarrow 0$. Therefore, we take limit in $\widehat{f_n * g} = \hat{f_n}\,\hat{g}$ to get $\widehat{f * g} = \hat{f}\,\hat{g}$.

Remember: to define $\hat{f}$ on L^2 it is sufficient to consider ANY SEQUENCE $f_n \in L^2 \cap L^1$ s.t. $f_n \to f$ in L^2.

T17 We apply Riesz–Thorin interpolation result. Namely, if

$$T: L^{p_0} \to L^{q_0}, \quad T: L^{p_1} \to L^{q_1} \qquad \forall_{\theta \in [0,1]} \quad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0}$$

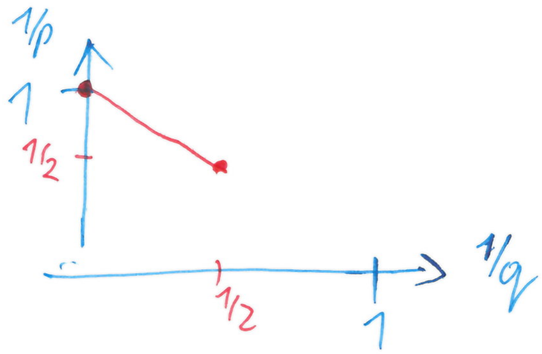
Then $T: L^{p_\theta} \to L^{q_\theta}$, $\|T\| \le \|T\|^{\theta}_{p_0,q_0} \|T\|^{1-\theta}_{p_1,q_1}$.

We have $\mathcal{F}: L^2 \to L^2$, $\mathcal{F}: L^1 \to L^\infty$.

$$\begin{cases} \dfrac{1}{p_\theta} = \dfrac{\theta}{2} + \dfrac{1-\theta}{1} \Rightarrow \dfrac{1}{p_\theta} = \dfrac{1}{q_\theta} + 1 - \dfrac{2}{q_\theta} \Rightarrow \boxed{\dfrac{1}{p_\theta} + \dfrac{1}{q_\theta} = 1} \\[2ex] \dfrac{1}{q_\theta} = \dfrac{\theta}{2} + \dfrac{1-\theta}{\infty} \Rightarrow \dfrac{1}{q_\theta} = \dfrac{\theta}{2} \end{cases}$$

$p_\theta \in [1,2]$, $q_\theta \in [2,\infty)$.

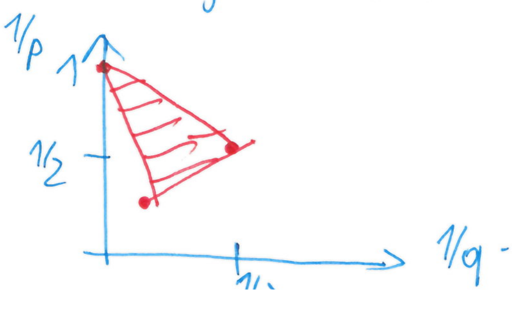
Therefore, $\mathcal{F}: L^p \to L^{p'}$, $p \in [1,2]$. People in Harmonic Analysis draw it as follows:



$$\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\left(\frac{1}{p}, \frac{1}{q}\right) = (1, 0)$$

The advantage of this picture is as follows: if one knows the third admissible pair of exponents $(\tilde{p}, \tilde{q})$, then by R-T interpolation the whole triangle is admissible.



Invitation to the theory of tempered distributions.

- Map $T: S(\mathbb{R}^n) \to \mathbb{C}$ is a temp. distr. if it is linear and

  $Tf_n \to Tf$ whenever $f_n \to f$ in $S(\mathbb{R}^n)$. We call this property boundedness of T. Space of all such T is denoted with $S'(\mathbb{R}^n)$.

- Topology on $S'(\mathbb{R}^n)$ is introduced with pointwise convergence: if $T_n(f) \to T(f)$ for all $f \in S(\mathbb{R}^n)$ then we say $T_n \to T$ in $S'(\mathbb{R}^n)$.

- Motivation: Fourier transform on L^1, L^2 and S is a very convenient tools but most of its properties require additional technical assumption ($\partial_x f \in L^1$ etc). Tempered distributions device allows not to worry too much about these assumptions.

- Fourier transform on $S'(\mathbb{R}^n)$ if $T \in S'(\mathbb{R}^n)$ we write $\hat{T} \in S'(\mathbb{R}^n)$ for $\hat{T}(f) := T(\hat{f})$.

TD 1 Fourier transform is continuous on $S'(\mathbb{R}^n)$: if $T_n \to T$ in $S'(\mathbb{R}^n)$ then $\hat{T}_n(f) = T_n(\hat{f}) \longrightarrow T(\hat{f}) = \hat{T}(f)$ as desired.

Isomorphism: the inverse is defined as $\check{T}(f) = T(\check{f})$.

Idea: One quickly sees that this way we move all regularity problems from $S'(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$.

TD2 Any $L^p(\mathbb{R}^n)$ function defines a temp. dist. Indeed, let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. We define $I_f(g) = \int_{\mathbb{R}^n} f(x) g(x)$, $g \in S(\mathbb{R}^n)$.

Clearly, I_f is linear. Let $g_n \to g$ in $S(\mathbb{R}^n)$. We need to check that $I_f(g_n) \to I_f(g)$.

$$\left| \int_{\mathbb{R}^n} f(x) \left(g(x) - g_n(x) \right) dx \right| \leq \|f\|_p \underbrace{\|g - g_n\|_{p'}} \longrightarrow 0.$$

$\to 0$: take sufficiently high polynomial seminorm

Indeed, we checked that any L^p norm is bdd with $S(\mathbb{R}^n)$ seminorm $p_{\alpha,\beta}$ for appropriate $\alpha, \beta \in \mathbb{N}^d$.

This allows to define Fourier transform of any L^p fun, $1 \leq p \leq \infty$.

TD3 Let $f \in L^1(\mathbb{R}^n)$. Consider it as an element of $S'(\mathbb{R}^n)$, i.e.

$I_f(g) = \int_{\mathbb{R}^n} f(x) g(x)$, $g \in S(\mathbb{R}^n)$. Then $\widehat{I_f}(g) = \int_{\mathbb{R}^n} f(x) \hat{g}(x)\, dx =$

$$= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} g(y) e^{-2\pi i y \cdot x}\, dy\, dx = \int_{\mathbb{R}^n} \hat{f}(y) g(y)\, dy$$

here, we used $f \in L^1(\mathbb{R}^n)$.

On the other hand, $\hat{f}$ defines a tempered distribution $I_{\hat{f}}(g) = \int_{\mathbb{R}^n} \hat{f}(y) g(y)\, dy$ so that $\boxed{\widehat{I_f}(g) = I_{\hat{f}}(g)}$ for $f \in L^1(\mathbb{R}^n)$. In particular, if $T \in S(\mathbb{R}^n)$, then $\hat{T}$ is not just a functional if $\exists$ function t s.t. $\hat{T}(g) = I_t(g)$.

In particular, we see that the new approach extends the previous definitions ($L^1, L^2, S \ldots$).

TD 4 Price: If $f \in L^p$, $\widehat{I_f}$ is just a functional on $S(\mathbb{R}^n)$ and not necessarily a function (i.e. not always we'll be able to find some F such that $\widehat{I_f}(g) = I_F(g)$).