Functional Analysis : PROBLEM SET 1

(normed spaces, Banach spaces, L^p spaces...)

- Review normed space, Banach space, convergenec in norm...
- Review L^p space, L^p norm,

L1 ⇑ This is homework assignment (in fact, case $i=2$ was discussed in the lectures of Analysis II, case $i>2$ follows by induction).

Some comment: When one studies inequality it is worth remembering how ONE CAN USE THIS INEQUALITY. Here one can bound p norm with p_1 and p_2 norms s.t. $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Note that $\frac{1}{p} \geq \frac{1}{p_1}, \frac{1}{p_2}$ i.e. $p_1, p_2 \geq p$ so that we always need to know "more integrability"!

L2 To prove that $(L^p(X,\mathcal{F},\mu), \|\cdot\|_p)$ is a normed space one needs Minkowski inequality (to check that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$). Trick here goes like this:

$$\int |f+g|^p \leq \int |f+g|^{p-1} \overbrace{|f+g|}^{\leq |f|+|g|} \underset{\text{Hölder}}{\leq} \left(\int |f+g|^{(p-1)p'} \right)^{1/p'} \left[\left(\int |f|^p \right)^{1/p} + \left(\int |g|^p \right)^{1/p} \right]$$

where $1 = \frac{1}{p} + \frac{1}{p'} \Rightarrow p' = \frac{p}{p-1} \Rightarrow \|f+g\|_p^p \leq \|f+g\|_p^{p-1} \left(\|f\|_p + \|g\|_p \right)$

As $\|f+g\|_p$ is finite (why?), we can divide concluding the proof.

Similarly, we can prove GENERALIZED MINKOWSKI INEQUALITY:

$$\left| \int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^p d\nu(y) \right| \leq \left| \int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^{p-1} \left| \int_X F(z,y)\, d\mu(z) \right| d\nu(y) \right|$$

$$\leq \left| \int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^{p-1} \int_X |F(z,y)|\, d\mu(z)\, d\nu(y) \right| =$$

$$= \int_X \int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^{p-1} |F(z,y)|\, d\nu(y)\, d\mu(z) \leq$$

Hölder for integral in y

$$\leq \int_X \underbrace{\left(\int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^p d\nu(y) \right)^{\frac{p-1}{p}}}_{\text{first part of Hölder estimate}} \underbrace{\left(\int_Y |F(z,y)|^p \, d\nu(y) \right)^{1/p}}_{\text{second part of Hölder estimate}} d\mu(z)$$

If we knew that $\int_Y \left| \int_X F(x,y)\, d\mu(x) \right|^p d\nu(y)$ is finite, we could divide and conclude the proof. But in general, we <u>don't know</u>. This is <u>IMPORTANT</u> technical point. We replace $F(x,y)$ with $F(x,y)\, \mathbb{1}_{\{|F| \leq n\}} \mathbb{1}_{\{X_n\}} \mathbb{1}_{\{Y_n\}} = F_n$

sets from σ-finiteness

and apply Monotone Convergence Theorem <u>(if $F \geq 0$)</u>.

(L3) (standard application of Hölder) Let $p \geq q$. Then

$$\|f\|_q^q = \int |f|^q = \int |f|^q \cdot 1 \leq \left(\int |f|^p \right)^{q/p} \left(\int 1 \right)^{p-q/p}$$

Hölder with $\frac{p}{q} \rightarrow 1 = \frac{1}{\left(\frac{p}{q}\right)} + \frac{1}{?} \Rightarrow \frac{1}{?} = \frac{p-q}{p}$

$$\Rightarrow \|f\|_q \leq \|f\|_p \cdot [\mu(X)]^{p-q/pq}$$

Note that $\frac{1}{x} \in L^2(1,\infty)$ but $\frac{1}{x} \notin L^1(1,\infty)$.

(L4) (exercise in Topology) $Y \subset X$, $(X, \|\cdot\|_X)$ Banach space

$(Y, \|\cdot\|_X)$ is Banach $\iff$ Y is closed in $(X, \|\cdot\|_X)$.

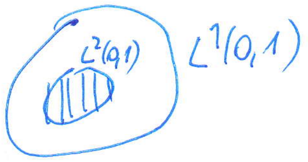
($\Leftarrow$) Let $(y_n) \subset Y$ be Cauchy sequence. Then, since $Y \subset X$, it is also Cauchy in X and since it is Banach space, it has limit in X. Call it $x \in X$. By closedness of Y, $x \in Y$ and assertion follows.

($\Rightarrow$) Let $(y_n) \subset Y$ be converging sequence, i.e. $y_n \rightarrow x$ in $\|\cdot\|_X$ for some $x \in X$. Since any converging sequence is Cauchy, $x \in Y$. $\square$

This fact is quite simple but allows to handle many Banach spaces.

L5 (standard app. of L4)

Observe that by L3, set $L^2(0,1)$ is subset of $L^1(0,1)$. Therefore,



if $(L^2(0,1), \|\cdot\|_1)$ were Banach space, $L^2(0,1)$ would be closed in $(L^1(0,1), \|\cdot\|_1)$.

The point is that there are fcns f in L^1 but not in L^2, for instance $\frac{1}{\sqrt{x}} \in L^1(0,1)$ but $\frac{1}{\sqrt{x}} \notin L^2(0,1)$. To be more precise, let

$$f_n(x) = \min\left(\frac{1}{\sqrt{x}}, n\right)$$

Claim: $f_n \to \frac{1}{\sqrt{x}}$ in L^1. $\iff \int \left| f_n - \frac{1}{\sqrt{x}} \right| \to 0$ by dominated convergence. This contradicts closedness of L^2 in $(L^1, \|\cdot\|_1)$.

In general case, we consider $\frac{1}{x^{1/p}}$ instead of $\frac{1}{x^{1/2}}$.

L8 (very important!!!)

X Banach space $\iff \sum_{k=1}^{\infty} \|x_k\| < \infty \Rightarrow \sum_{k=1}^{\infty} x_k$ converges in X

($\Leftarrow$) Let (x_k) be a Cauchy sequence in X. We only need to check that it has convergent subsequence (exercise in Topology ...)

Choose subsequence s.t. $\|x_{k_{n+1}} - x_{k_n}\| \leq 2^{-k}$. $\Rightarrow \sum \|x_{k_{n+1}} - x_{k_n}\| < \infty$

$\Rightarrow \sum x_{k_{n+1}} - x_{k_n}$ converges in X $\Rightarrow \{x_{k_n}\}$ converges in X $\Rightarrow$ $\{x_n\}$ converges in X. (as this is Cauchy sequence).

($\Rightarrow$) it is easy as $\left(\sum_{k=1}^{n} x_k\right)_{n \in \mathbb{N}}$ is a Cauchy sequence .... $\square$.

We used here simple lemma from Topology: if $\{x_n\}$ is Cauchy and it has convergent subsequence then the whole sequence converges. Indeed, let $\varepsilon > 0$, $x_{n_k} \to x$ be convergent subsequence. There is N s.t. $\forall_{n,m \geq N}$ $\|x_n - x_m\| \leq \frac{\varepsilon}{2}$. There is also M s.t. $n_k \geq M$ $\|x_{n_k} - x\| \leq \varepsilon/2$.

Take $\max(M, N)$ $\square$.

(L9) L^p spaces are complete once again. $(1 \leq p < \infty)$

Let $\sum \|f_n\|_p < \infty$. We need to prove that $\sum_{n=1}^{\infty} f_n$ converges in L^p. First, let us study $\sum_{n=1}^{\infty} |f_n|$. By Fatou and Minkowski,

$$\sum_{n=1}^{\infty} |f_n| = g \qquad g_k = \sum_{n=1}^{k} |f_n|$$

$$\left(\int |g|^p\right)^{1/p} = \left(\int \lim_{n\to\infty} |g_n|^p\right)^{1/p} \leq \liminf_{n\to\infty} \left(\int |g_n|^p\right)^{1/p} \leq \sum \|f_n\|_p < \infty$$

so $g \in L^p$. In particular, $g < \infty$ a.e. in X and $\sum f_n$ converges absolutely to some function (defined pointwisely ...). Call this function f. By Dominated Convergence, $\sum f_n \to f$ in L^p.

(L6) ↑↑ (Homework assignment)

(S1) This is application of results for L^p with counting measure. Indeed, then

$$\int |f|^p d\mu = \sum |f_i|^p \quad (\text{for } f = (f_1, f_2, \dots)). \quad \left(\mu \text{ - counting measure}\right) \square.$$

(S2) (important!!!) → this exercise allows to deduce many properties from considering finite sequences.... For $1 \leq p < \infty$:

$$x - \sum_{i=1}^{n} x_i e_i = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$$

$$\|x - \sum_{i=1}^{n} x_i e_i\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \to 0 \text{ as } n\to\infty$$ since the series $\sum_{k=1}^{\infty} |x_k|^p$ is convergent (this is tail of convergent series).

Case $p=\infty$ is NOT true. Take for instance $x = (1, 1, 1, \dots, 1, \dots) \in l^\infty$. (sequence of 1). Then $x - \sum_{i=1}^{n} x_i e_i = (0, 0, 0, \dots, 0, 1, 1, \dots)$ so its norm in l^∞ is 1 for all n.

(A1)

(a) $\rho(\alpha x) = \inf\{\lambda > 0 : \frac{\alpha x}{\lambda} \in C\} = \inf\{\lambda > 0 : \frac{x}{\underbrace{\lambda/\alpha}_{\gamma}} \in C\}$ $\quad \lambda = \gamma\alpha$

$= \inf\{\gamma\alpha : \frac{x}{\gamma} \in C\} = \alpha \cdot \inf\{\gamma : \frac{x}{\gamma} \in C\}.$

(b) As ρ is defined as infimum, it would be sufficient to check that $\frac{x+y}{\rho(x)+\rho(y)} \in C$. We can write $\frac{x+y}{\rho(x)+\rho(y)} = \frac{x}{\rho(x)} \frac{\rho(x)}{\rho(x)+\rho(y)} + \frac{y}{\rho(y)} \cdot \frac{\rho(y)}{\rho(x)+\rho(y)}$

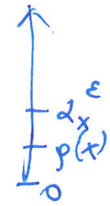
= Convex combination of $\frac{x}{\rho(x)}, \frac{y}{\rho(y)}$. But: we don't know $\frac{x}{\rho(x)}, \frac{y}{\rho(y)} \in C$

But: $\rho(x), \rho(y)$ are inf of α_x, α_y and $\frac{x}{\alpha_x}, \frac{y}{\alpha_y} \in C$.

So the good solution goes as follows:

Let $\varepsilon > 0$. Let $\alpha_x^\varepsilon, \alpha_y^\varepsilon$ be s.t. $\frac{x}{\alpha_x^\varepsilon}, \frac{y}{\alpha_y^\varepsilon} \in C$ and

$\rho(x) + \varepsilon \geq \alpha_x^\varepsilon, \quad \rho(y) + \varepsilon \geq \alpha_y^\varepsilon.$



Then

$\frac{x+y}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} = \frac{x}{\alpha_x^\varepsilon} \cdot \frac{\alpha_x^\varepsilon}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} + \frac{y}{\alpha_y^\varepsilon} \cdot \frac{\alpha_y^\varepsilon}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} \in C$ by convexity.

$\Rightarrow \rho(x+y) \leq \alpha_x^\varepsilon + \alpha_y^\varepsilon \leq \rho(x) + \rho(y) + 2\varepsilon$. As $\varepsilon > 0$ is arbitrary $\Rightarrow \rho(x+y) \leq \rho(x) + \rho(y)$. $\square$.

(c) Clearly, $\rho(x) \geq 0$. As $0 \in C$, C is open, there is r s.t. $B(0,r) \subset C$. But then $\frac{x}{\|x\|} \cdot \frac{r}{2} \in C \Rightarrow \frac{x}{\|x\|} \cdot \frac{r}{2} = \frac{x}{\|x\| \cdot \frac{2}{r}} \in C \Rightarrow$

$\rho(x) \leq \|x\| \cdot \frac{2}{r}$, take $M = \frac{2}{r}$. $\square$.

(d) $C = \{\rho(x) < 1\}$.

$\supseteq$: as $\rho(x) < 1$, there is α s.t. $\frac{x}{\alpha} \in C$ and $\alpha < 1$.

But then $x = \alpha \cdot \frac{x}{\alpha} + (1-\alpha) \cdot 0 \in C$ by convexity.

$\subseteq$: As $x \in C$ and C open there is $\varepsilon > 0$ s.t. $(1+\varepsilon)x \in C$, $(1+\varepsilon)x = \frac{x}{\frac{1}{1+\varepsilon}} \Rightarrow$

$\varrho(x) \leq \frac{1}{1+\varepsilon} < 1 \quad \Rightarrow \quad \varphi(x) < 1$ as desired.

$\square$.

C3 ⇑

In the lecture: $C([0,1])$ with $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$ is Banach space. Here, we construct subspace that is <u>not</u> Banach with inherited norm.

S3

c is Banach space

It is sufficient to check c is closed in ℓ^∞ which is Banach.

Let x^k be sequence such that $x^k \xrightarrow{\ell^\infty} x$, $\{x^k\} \subset c$, $x \in \ell^\infty$. We have to check $x \in c$. (We write $x = (x_n)_{n \geq 1}$).

~~We know $\sup_n |x_n^k - x_n| \to 0$, in particular $\forall_n$ $|x_n^k - x_n| \to 0$~~

Note that x^k is (in particular) Cauchy in ℓ^∞, i.e.

$$\forall_{\varepsilon > 0} \ \exists_N \ \forall_{k_1, k_2 \geq N} \ \sup_n |x_n^{k_1} - x_n^{k_2}| \leq \varepsilon$$

i.e. $$\forall_{\varepsilon > 0} \ \exists_N \ \forall_{k_1, k_2 \geq N} \ \forall_n \ |x_n^{k_1} - x_n^{k_2}| \leq \varepsilon. \qquad (*)$$

As this is satisfied for all n, we pass to the limit $((x^k) \subset c)$ with $n \to \infty$ to have

$$\forall_{\varepsilon > 0} \ \exists_N \ \forall_{k_1, k_2 \geq N} \ |x^{(k_1)} - x^{(k_2)}| \leq \varepsilon$$

where $x^{(k_1)}$, $x^{(k_2)}$ are limits $\lim_{n\to\infty} x_n^{k_1}$, $\lim_{n\to\infty} x_n^{k_2}$. In particular, limits are Cauchy $\Rightarrow$ there is $x^{(\infty)} \in \mathbb{R}$ s.t. $\lim_{k\to\infty} x^{(k)} = x^{(\infty)}$.

We claim that $\lim_{n\to\infty} x_n = x^{(\infty)}$ and so $x \in c$. Indeed, $\forall_k$

$$|x^{(\infty)} - x_n| \leq |x^{(\infty)} - x^{(k)}| + |x^{(k)} - x_n^{(k)}| + |x_n^{(k)} - x_n|$$

~~$\Rightarrow \limsup_{n\to\infty} |x^{(\infty)} - x_n| \leq |x^{(\infty)} - x^{(k)}| +$, the last term vanishes by passing with $k \to \infty$ in~~

For any k, $\limsup_{n\to\infty} |x^{(k)} - x_n^{(k)}| = 0$ since $x^{(k)} = \lim_{k\to\infty} x_n^{(k)}$.

Moreover, let $\varepsilon > 0$. By $(*)$ $\exists_N \ \forall_{k_1,k_2 \geq N} \ \forall_n \ |x_n^{k_1} - x_n^{k_2}| \leq \varepsilon$.

Send $k_2 \to \infty \Rightarrow \exists_N \ \forall_{k_1 \geq N} \ \forall_n \ |x_n^{k_1} - x_n| \leq \varepsilon$

Send $n \to \infty \Rightarrow \exists_N \ \forall_{k_1 \geq N} \ |x^{(k_1)} - x^{(\infty)}| \leq \varepsilon$

$\Rightarrow \limsup_{n\to\infty} |x^{(\infty)} - x_n| \leq 2\varepsilon$ so $x_n \to x^{(\infty)}$. $\square$.

S4 c_0 is Banach space (homework) [Introduce this space c_0 but Banach $\to$ HW]

S5 This is like in ℓ^p $(1 \leq p < \infty)$. We have

$x - \sum_{i=1}^{n} x_i e_i = (\underbrace{0, 0, \dots, 0}_{n}, x_{n+1}, x_{n+2}, \dots)$ so $\|x - \sum_{i=1}^{n} x_i e_i\|_\infty =$

$\sup_{k \geq n+1} |x_k| \to 0$ as $x \in c_0$.

S6 It can't – it's not linear.

S7 homework.

A1

(a) $\rho(\alpha x) = \inf\left\{ \lambda > 0 : \frac{\alpha x}{\lambda} \in C \right\} = \inf\left\{ \lambda > 0 : \frac{x}{\underbrace{\frac{\lambda}{\alpha}}_{\gamma}} \in C \right\}$ $\quad \lambda = \gamma\alpha$

$= \inf\left\{ \gamma\alpha : \frac{x}{\gamma} \in C \right\} = \alpha \cdot \inf\left\{ \gamma : \frac{x}{\gamma} \in C \right\}.$

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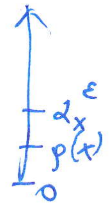
= Convex combination of $\frac{x}{\rho(x)}, \frac{y}{\rho(y)}$. <u>But</u>: we don't know $\frac{x}{\rho(x)}, \frac{y}{\rho(y)} \in C$

<u>But</u>: $\rho(x), \rho(y)$ are inf of α_x, α_y and $\frac{x}{\alpha_x}, \frac{y}{\alpha_y} \in C$.

So the good solution goes as follows:

Let $\varepsilon > 0$. Let $\alpha_x^\varepsilon, \alpha_y^\varepsilon$ be s.t. $\frac{x}{\alpha_x^\varepsilon}, \frac{y}{\alpha_y^\varepsilon} \in C$ and

$\rho(x) + \varepsilon \geq \alpha_x^\varepsilon, \quad \rho(y) + \varepsilon \geq \alpha_y^\varepsilon.$



Then

$$\frac{x+y}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} = \frac{x}{\alpha_x^\varepsilon} \cdot \frac{\alpha_x^\varepsilon}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} + \frac{y}{\alpha_y^\varepsilon} \cdot \frac{\alpha_y^\varepsilon}{\alpha_x^\varepsilon + \alpha_y^\varepsilon} \in C \text{ by convexity.}$$

$\Rightarrow \rho(x+y) \leq \alpha_x^\varepsilon + \alpha_y^\varepsilon \leq \rho(x) + \rho(y) + 2\varepsilon$. As $\varepsilon > 0$ is arbitrary $\Rightarrow \rho(x+y) \leq \rho(x) + \rho(y)$. $\square$.

(c) Clearly, $\rho(x) \geq 0$. As $0 \in C$, C is open, there is r s.t. $B(0,r) \subset C$. But then $\frac{x}{\|x\|} \cdot \frac{r}{2} \in C \Rightarrow \frac{x}{\|x\|} \cdot \frac{r}{2} = \frac{x}{\|x\| \cdot \frac{2}{r}} \in C \Rightarrow$

$\rho(x) \leq \|x\| \cdot \frac{2}{r}$, take $M = \frac{2}{r}$. $\square$.

(d) $C = \{ \rho(x) < 1 \}$.

$\supseteq$: as $\rho(x) < 1$, there is α s.t. $\frac{x}{\alpha} \in C$ and $\alpha < 1$.

But then $x = \alpha \cdot \frac{x}{\alpha} + (1-\alpha) \cdot 0 \in C$ by convexity.

$\rho(x) \leq \frac{1}{1+\varepsilon} < 1$

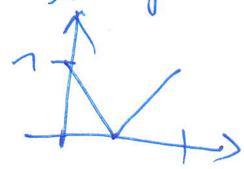
$\Uparrow$

$\subseteq$: As $x \in C$ and C open there is $\varepsilon > 0$ s.t. $(1+\varepsilon)x \in C$, $(1+\varepsilon)x = \frac{x}{\frac{1}{1+\varepsilon}} \Rightarrow$

(C2) Let f_n be Cauchy sequence in $C^1[0,1]$. Then f_n, f_n' are Cauchy in $C[0,1]$ so by completeness, there are f, g s.t. $f_n \to f$, $f_n' \to g$ uniformly. We have to check that $g = f'$.

Indeed, for $n \in \mathbb{N}$, $f_n(t) = f_n(0) + \int_0^t f_n'(s)\,ds$ and using uniform convergence, $f(t) = f(0) + \int_0^t g(s)\,ds \Rightarrow f' = g$ as desired. $\square$.

(C4) • $\|f\|_B$ is not a norm: $\|1\|_B = 0$

• $\|f\|_A$ is a norm on $C^1[0,1]$ as it is a norm on the bigger space $C[0,1]$. Suppose $(C^1[0,1], \|\cdot\|_A)$ is Banach. Let f be 

(i.e. $|x - \frac{1}{2}| \cdot 2$). Let p_n be a seq of polynomials s.t. $\|f - p_n\|_\infty^2 \to 0$.
Note that $p_n \in C^1([0,1])$ so we get contradiction.

(C5) (Big Homework)