Functional Analysis (WS 19/20), Problem Set 2

(operators and their norms)

Note: some facts (especially the general ones) should be discussed in the lecture and if that is the case, they can be skipped.

For the linear map $T : X \to Y$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces, we define norm of the operator T as

$$\|T\| = \sup_{x:\|x\|_X \leq 1} \|Tx\|_Y.$$

If $\|T\| < \infty$ we say T is a bounded linear operator. We write $\mathcal{L}(X, Y)$ for the set of all bounded linear operators between X and Y. We write X^* for the dual space of X i.e. space of bounded and linear operators $X : E \to \mathbb{R}$.

Some facts that should be discussed in the lecture

F1. Check that

$$\sup_{x:\|x\|_X \leq 1} \|Tx\|_Y = \sup_{x:\|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

so that there are three equivalent definition of the operator norm.

F2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $T : X \to Y$ be a linear map. Check that the following conditions are equivalent:

- (a) T is a bounded linear operator,
- (b) T is continuous at 0,
- (c) T is continuous,
- (d) T is Lipschitz continuous.

F3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Justify briefly that $\mathcal{L}(X, Y)$ equipped with operator norm is a normed space.

F4. Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Prove that $\mathcal{L}(X, Y)$ equipped with operator norm is a Banach space. **Note**: it is sufficient that $(Y, \|\cdot\|_Y)$ is a Banach space.

Some additional facts about operators

A1. (**dual spaces are always Banach**) Let $(X, \|\cdot\|_X)$ be a normed space. Justify briefly that the dual space X^* of bounded linear functionals on X is a Banach space.

A2. **(unbounded functionals)** Let $(X, \|\cdot\|_X)$ be infinite dimensional Banach space. Prove that there is a linear functional that is not bounded. *Hint*: consider Hamel basis of X and define the functional on its elements. *Remark:* It can get even worse: "projection" functionals can be discontinuous. To see this consider space of polynomials with L^1 norm and study projection of $(x-1)^n$ on 1 which is an element of Hamel basis.

A3. **(kernel of bounded functionals)** Let $(X, \|\cdot\|_X)$ be a Banach space and let φ be a linear functional on X. Prove that $\varphi \in X^*$ if and only if its kernel is closed. Prove also that the kernel of a bounded linear functional $\varphi \in X^*$ has codimension 1.

A4. **(extension from dense subset)** Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Suppose that D is a dense linear subspace of X and $T : D \to Y$ is a bounded linear operator. Prove that T has a unique extension to X which preserves the norm.

A5. **(bound on composition)** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Let $T : X \to Y$ and $S : Y \to Z$ be bounded linear operators. Check that $\|S \circ T\| \leq \|S\| \, \|T\|$.

A6. (**exponential of the operator**) Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Let $T : X \to Y$ be a bounded operator. Check that the series $\sum_{k=0}^{\infty} \frac{T^k}{k!}$ converges in $\mathcal{L}(X, Y)$. This is usually definition of e^T. Note that this generalizes e^A for $A \in \mathbb{R}^{n \times n}$.

Introduction to invertibility - norm condition

I1. (**necessary condition for series convergence**) Let $(X, \|\cdot\|_X)$ be a normed space. Suppose that $\sum_{k=0}^{\infty} x_k$ converges in $(X, \|\cdot\|_X)$. Prove that $x_k \to 0$.

I2. Let $(X, \|\cdot\|_X)$ be a normed space and $T \in \mathcal{L}(X, X)$. Prove that if $\sum_{k=1}^{\infty} T^k$ converges in $\mathcal{L}(X, X)$ then $(I - T)^{-1}$ exists and

$$(I - T)^{-1} = \sum_{k=1}^{\infty} T^k.$$

Moreover, if $(X, \|\cdot\|_X)$ is a Banach space, it is sufficient that $\sum_{k=1}^{\infty} \|T\|^k < \infty$, i.e. $\|T\| < 1$.

I3. Let $k \in C([0, 1] \times [0, 1])$ with $\|k\|_\infty < 1$ and $y \in C([0, 1])$. Prove that there is a unique continuous solution $x \in C([0, 1])$ to the integral equation

$$x(t) - \int_0^1 k(s, t) x(s) \, ds = y(t).$$

I4. Let $(X, \|\cdot\|_X)$ be a Banach space. Prove that the set of invertible operators is open in $\mathcal{L}(X, X)$ equipped with operator norm. *Hint:* Consider ball in $\mathcal{L}(X, X)$ centered at T. If S is in that ball, write $S = T + W = T(I + T^{-1}W)$ for some "small" W. **Note**: Any bounded linear operator that is invertible can be perturbed (in a sufficiently small way) and the resulting perturbation is still invertible.

Standard problems – computation of the operator norms

It can be considered as a tradition of the Faculty that on the mid-term exam there is a question of the type "compute norm of the operator". Therefore, below I have compiled some list of problems to practice.

N1. Let $E = \{u \in C([0, 1]) : u(0) = 0\}$. Justify briefly that E with a supremum norm is a Banach space. Consider a linear functional $\varphi : E \to \mathbb{R}$ defined with $\varphi(u) = \int_0^1 u(t) \, dt$. Compute norm $\|\varphi\|$ of this functional (i.e. $\sup_{u:\|u\|_\infty \leq 1} |\varphi(u)|$) and prove that $\varphi \in E^*$. Is there $u \in E$ such that $\|u\|_\infty \leq 1$ and $\varphi(u) = \|\varphi\|$?

N2. Consider a linear functional $\varphi : C[0, 1] \to \mathbb{R}$ defined with $\varphi(u) = \int_0^{\frac{1}{2}} u(t) \, dt - \int_{\frac{1}{2}}^1 u(t) \, dt$. Compute norm $\|\varphi\|$ of this functional (i.e. $\sup_{u:\|u\|_\infty \leq 1} |\varphi(u)|$) and prove that $\varphi \in (C[0, 1])^*$. Is there $u \in C[0, 1]$ such that $\|u\|_\infty \leq 1$ and $\varphi(u) = \|\varphi\|$?

N3. Consider functional $\varphi : c_0 \to \mathbb{R}$ defined with $\varphi(u) = \sum_{i=1}^{\infty} \frac{1}{2^n} u_n$ where $u = (u_1, u_2, u_3, ...)$. Compute norm $\|\varphi\|$ of this functional (i.e. $\sup_{u:\|u\|_\infty \leq 1} |\varphi(u)|$) and prove that $\varphi \in (c_0)^*$. Is there $u \in c_0$ such that $\|u\|_\infty \leq 1$ and $\varphi(u) = \|\varphi\|$?

N4. Check if the following formulas define bounded linear functionals on given spaces (and if yes, compute their norm):

(a) $\varphi(a) = \lim_{k\to\infty} a_k$ on c (space of convergent sequences),

(b) $\varphi(f) = \int_{-1}^{-1} x f(x)\, dx$ on $C[-1, 1]$.

N5. Consider operator $T : l_1 \to c_0$ defined with

$$(Ty)_k = \frac{1}{2} \sum_{j=1}^{k} \left(\frac{1}{k}\right)^j y_j,$$

where $y = (y_1, y_2, ...)$ and $(Ty)_k$ is k-th element of sequence Ty. Check that T is well-defined (i.e. $Ty \in c_0$ whenever $y \in l_1$), prove that it is a bounded linear operator and compute its norm.

N6. Consider operator $T : c_0 \to l^\infty$ given by $Tx = \left(\sum_{j\geq 1} a_{i,j} x_j\right)_{i=1}^{\infty}$ where numbers $a_{i,j}$ are such that $\sup_{i\geq 1} \sum_{j\geq 1} |a_{i,j}| < \infty$. Prove that $\|T\| = \sup_{i\geq 1} \sum_{j\geq 1} |a_{i,j}|$.

N7. (**averaging operators**) Let $1 \leq p \leq \infty$ and $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be defined with

$$T(f)(y) = \frac{1}{\lambda_n(B(y, 1))} \int_{B(y,1)} f(x)\, dx$$

where $B(y, 1)$ denotes a unit ball with center $y \in \mathbb{R}^d$. Check that T is well - defined (i.e. it is L^p-valued), prove that it is a bounded linear operator and compute its norm.

N8. (**discrete derivative**) Let $1 \leq p \leq \infty$ and $T : l^p \to l^p$ be defined with

$$T((a_n)_{n\geq 1}) = (a_{n+1} - a_n)_{n\geq 1}.$$

Check that T is well - defined (i.e. it is l^p-valued), prove that it is a bounded linear operator and compute its norm.

N9. Let $1 \leq p < \infty$ and consider operator $T : l^p \to l^p$ defined with

(a) $T((a_n)_{n\geq 1}) = \left(\frac{a_n}{n+1}\right)_{n\geq 1}$,

(b) $T((a_n)_{n\geq 1}) = (a_{2n} + a_n)_{n\geq 1}$.

Decide whether T is a well-defined bounded operator and if yes, compute its norm.