Functional Analysis : PROBLEM SET 2

(operators and their norms)

F1 $A = \sup_{\|x\|=1} \|Tx\|_Y$, $B = \sup_{\|x\|\leq 1} \|Tx\|_Y$, $C = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$

We know $B \geq A$, $C = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x \neq 0} \left\| T\left(\frac{x}{\|x\|_X}\right) \right\|_Y =$

$= \sup_{\|x\|=1} \|Tx\|_Y = A$.

Finally $B = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|\leq 1} \left\| \|x\| \, T\left(\frac{x}{\|x\|}\right) \right\|_Y =$

$= \sup_{\|x\|\leq 1} \|x\| \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \sup_{\|x\|\leq 1} \left\| T\left(\underbrace{\frac{x}{\|x\|}}_{\text{norm } 1.}\right) \right\| \leq A.$

F2 (d) $\Rightarrow$ (c) $\Rightarrow$ (b) easy

(a) $\Rightarrow$ (d): $\|T(x-y)\| = \left\| T \frac{(x-y)}{\|x-y\|} \right\| \overbrace{}^{\|x-y\|} \leq \|T\| \, \|x-y\|$
so that T is Lipschitz

(b) $\Rightarrow$ (a): suppose T is not bdd : $\exists\, x_n$, $\|x_n\|=1$ but $\|Tx_n\| \geq n$

Consider $y_n = \frac{x_n}{n}$ so that $y_n \to 0$ in X. But then

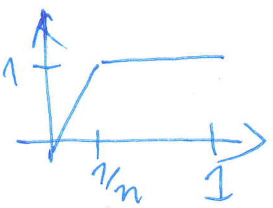
$\|Ty_n\| = \left\| T \frac{x_n}{n} \right\| \geq \frac{1}{n} \|Tx_n\| \geq 1$ so T is not cont.
at 0 $\Rightarrow$ contradiction.

N1 We need to find $\sup_{\|u\|_\infty \leq 1, u \in E} |\varphi(u)|$. Typically, one first finds bound, then tries to check whether its optimal.

Clearly $|\varphi(u)| \leq \left| \int_0^1 u(t)\, dt \right| \leq \int_0^1 \|u\|_\infty\, dt = \|u\|_\infty$

so $\sup_{\substack{\|u\|_\infty \leq 1 \\ u \in E}} |\varphi(u)| \leq 1$.

We claim that $\sup_{\|u\|_\infty \leq 1, u \in E} |\varphi(u)| = 1$. Althoug $u(0)=0$ we can approximate with fcns



$= f_n$, $\|f_n\|_\infty \leq 1$, $f_n \in E$.

Note that $\varphi(f_n) = 1 - \frac{1}{n} + \frac{1}{2n} = 1 - \frac{1}{2n} \to 1$.

By the upper bound, $\|\varphi\| = 1$.

There is no $f \in E$, $\|f\|_\infty = 1$ and $\varphi(f) = \|f\|_\infty$ as $f(0)=0$ and f has to be cont!

N2 Homework

N3 Upper bound: $|\varphi(u)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |u_n| \leq \|u\|_\infty \sum_{n=1}^{\infty} \frac{1}{2^n} = \|u\|_\infty$

Again, there is no $u \in C_0$ s.t. $\|u\|_\infty = 1$, $|\varphi(u)| = 1$ since

then $1 = \sum \frac{1}{2^n} |u_n| \leq 1$ so $u_n = 1$ $\forall n$.

But consider u^k s.t. $u^k = (\underbrace{1, 1, \dots, 1}_{k}, 0, 0, \dots) \in C_0$

and $\|u^k\|_\infty = 1$ and $\varphi(u^k) \to 1$.

$\Rightarrow \|\varphi\| = 1$. $\square$

(F3) Pretty standard, check that the $\|T\| = \sup_{\|x\|_X \leq 1} \|Tx\|_Y$ is a norm ...

(F4) (important!)

Let $\{A_n\} \in \mathcal{L}(X, Y)$ be a Cauchy sequence. Note that $\forall_x \{A_n x\}$ is a Cauchy sequence since

$$A_n \text{ Cauchy in } \mathcal{L}(X,Y) \Rightarrow \sup_{\|x\| \neq 0} \frac{\|(A_n - A_m)x\|_Y}{\|x\|_X} \leq \varepsilon \quad (*)$$

$\left(\forall_{\varepsilon > 0} \; \exists_N \; \forall_{n,m \geq N} \right)$. But this means that $\forall_{\varepsilon > 0} \; \exists_N \; \forall_{n,m} \; \forall_x$

$$\|A_n x - A_m x\|_Y \leq \varepsilon \|x\|_X.$$

Therefore $\{A_n x\}$ is convergent for all $x \in X$. We define

$$Ax = \lim_{n \to \infty} A_n x.$$

- A is linear: $A(x+y) = \lim_{n \to \infty} A_n(x+y) = \lim_{n \to \infty} A_n x + \lim_{n \to \infty} A_n y$

$$= Ax + Ay$$

- A is bounded $\|Ax\| \leq \sup \|A_n\| \|x\|$ so $\|A\| \leq \sup_n \|A_n\|$ which is finite since Cauchy sequences are bounded.

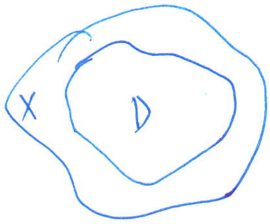
We need $A_n \to A$ in $\mathcal{L}(X, Y)$. We pass with $m \to \infty$ in $(*)$ to have $\forall_{\varepsilon > 0} \; \exists_N \; \forall_{n \geq N} \; \sup_{\|x\| \leq 1} \|A_n x - Ax\|_Y \leq \varepsilon$.

This concludes the proof.

$\square$.

(A1) For Banach space $(X, \|\cdot\|_X)$ we write $X^* = \mathcal{L}(X, \mathbb{R})$ so it is Banach space since $\mathbb{R}$ is complete (see F4).

(A4) (big homework)



extension by density – very important tool: usually operator cannot be defined on the whole X as func are not differentiable, integrable, etc...

(A5)
$$\|S \circ T\| = \sup_{x \neq 0 \text{ s.t. } Tx \neq 0} \|S \circ T x\|_Z / \|x\|_X =$$

$$= \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{\|S \circ T x\|_Z}{\|Tx\|_Y} \cdot \frac{\|Tx\|_Y}{\|x\|_X} \leq \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{\|S \circ T x\|_Z}{\|Tx\|_Y} \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$\leq \sup_{y \neq 0} \frac{\|Sy\|_Z}{\|y\|_Y} \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \leq \|S\| \, \|T\|. \quad \checkmark$$

(A6) $\mathcal{L}(X, Y)$ – Banach space,

$\|T^k\| \leq \|T\|^k$ by A5,

$\sum \frac{T^k}{k!}$ converges in $\mathcal{L}(X, Y)$ since $\sum \frac{\|T^k\|}{k!} = e^{\|T\|}$ converges.

Applications: ODEs on Banach spaces (Stochastic PDEs).

This is generalization of e^T for $T: Y \to Y$.

#2 In finite dimensional setting any linear functional is bounded (and so, continuous). Indeed, $x = \sum x_i e_i$

- $\varphi(x) = \sum_{i=1}^{n} x_i \varphi(e_i)$

- $\sum |x_i| \leqslant C \|x\|$ as all norms are equivalent

$\Rightarrow \sup_{\|x\| \leqslant 1} |\varphi(x)| \leqslant \sup_{1 \leqslant i \leqslant n} |\varphi(e_i)| < \infty.$

This is not the case in infinite-dim setting. We put $\varphi(e_i) = i$ for countable subset $\{e_i\}$ of Hamel basis and 0 otherwise. (wlog $\|e_i\| = 1$). Then $\sup_{\|x\| \leq 1} |\varphi(x)| \geq \sup_i |\varphi(e_i)| = \infty.$

$\square$.

<u>Projection can be discontinuous</u>:

Consider space of polynomials with L^1 norm. Consider $(x-1)^n$ so its projection on 1 is $(-1)^n$. $(x-1)^n \longrightarrow 0$ in L^1 but $(-1)^n \not\longrightarrow$ converge (no limit...)

N7 Let $1 \leq p < \infty$, $C_d = |B(0,1)|$

$$\int_{\mathbb{R}^d} |Tf(y)|^p dy = \int_{\mathbb{R}^d} \left[\fint_{B(y,1)} f(x)\,dx \right]^p dy \underset{\substack{\uparrow \\ \text{change} \\ \text{of variables}}}{=} \int_{\mathbb{R}^d} \left[\fint_{B(0,1)} f(x+y)\,dx \right]^p dy$$

$$\underset{\text{Jensen}}{\leq} \int_{\mathbb{R}^d} \fint_{B(0,1)} |f(x+y)|^p dx\,dy \underset{\substack{\uparrow \\ \text{Fubini}}}{=} \fint_{B(0,1)} \underbrace{\int_{\mathbb{R}^d} |f(x+y)|^p dy}_{x\text{-fixed}}\, dx$$

$$= \fint_{B(0,1)} \|f\|_p^p \, dx = \|f\|_p^p$$

$$\Rightarrow \sup_{\substack{f \neq 0 \\ f \in L^p}} \frac{\|Tf\|_p}{\|f\|_p} \leq 1.$$ We claim that $\sup_{f \in L^p, f \neq 0} \frac{\|Tf\|_p}{\|f\|_p} = 1$.

Consider $f_n = \mathbb{1}_{B_n(0)}$. Note that $\|Tf_n\|_{L^p} \geq \|f_{n-2}\|_{L^p}$.

So
$$\frac{\|Tf_n\|_p}{\|f_n\|_p} \geq \frac{\|f_{n-2}\|_p}{\|f_n\|_p} = \frac{\left(C_d (n-2)^d\right)^{1/p}}{\left(C_d\, n^d\right)^{1/p}} = \left(1 - \frac{2}{n}\right)^{d/p} \longrightarrow 1$$

as $n \to \infty$.

Case $p = \infty$ is trivial. $\|Tf\|_\infty \leq \|f\|_\infty$ and value is attained for $f = 1$ on $\mathbb{R}^d$.

$\square$.

N8 $\rightsquigarrow$ Big Homework, similar to N7.

I1 In any metric space, convergent sequences are Cauchy.

Hence, $\forall_{\varepsilon>0} \exists_N \forall_{n,m\geq N} \|S_n - S_m\| \leq \varepsilon$ where $S_n = \sum_{i=1}^{n} x_i$.

Take $m=n+1 \Rightarrow \forall_{\varepsilon>0} \exists_N \forall_{n\geq N} \|x_n\| \leq \varepsilon \Rightarrow x_n \to 0$.

I2 We have to check bijectivity i.e. existence of left and right inverses.

- If $f: X \to X$ has left inverse $g: X \to X$ (i.e. $g(f(x)) = x$) then f is injective (as $f(x)=f(y) \Rightarrow x=y$).
- If $f: X \to X$ has right inverse $g: X \to X$ (i.e. $f(g(x)) = x$) then f is surjective (as $\exists_x \forall_y f(y) \neq x \underset{\uparrow y=g(x)}{\Rightarrow}$ contradiction).

We prove that $\sum_{k\geq 0} T^k$ is left and right inverse of $(I-T)$.

Let $S_k = \sum_{k=0}^{k} T^k$. Note that $S_k(I-T) = (I-T)S_k$. Therefore, for any $x \in X$:

$$x = Ix = \lim_{m\to\infty}(I - T^{m+1})x \overset{T^n\to 0}{=} \lim_{m\to\infty}(I - T + T - T^2 + T^2 - T^3 + \ldots - T^{m+1})x$$

$$= \lim_{m\to\infty}(I-T)S_m x \underset{\text{continuity}}{=} (I-T)\lim_{m\to\infty} S_m x \underset{\text{convergence of } \sum T^n}{=} (I-T)\left(\sum_{n=0}^{\infty} T^n\right)x.$$

Similarly, $\left(\sum_{n=0}^{\infty} T^n\right)(I-T)x = x$.

Hence $\sum_{n=0}^{\infty} T^n = (I-T)^{-1}$.

In case of BS: $(I-T)$ is invertible if $\|T\| < 1$. Then,

$$\|(I-T)^{-1}\| \leq \frac{1}{1-\|T\|}$$

□.

I3 We study operator $T : C[0,1] \to C[0,1]$ given with

$$(Tx)(t) = \int_0^1 k(s,t)\, x(s)\, ds \Rightarrow \|T\| \leq \|k\| < 1.$$

Hence $I - T$ is invertible $\Rightarrow x = (I-T)^{-1} y$ and has to be unique.

I4 Let $T \in \mathcal{L}(X,X)$. We want to find a ball in $\mathcal{L}(X,X)$ centered in T s.t. all operators in that ball are invertible (so we need to find radius of such ball). If S is in that ball we have

$$S = T + W = T(I + T^{-1}W) = T(I - (-T^{-1}W))$$

and this is invertible provided $\|T^{-1}W\| < 1$.

We estimate $\|T^{-1}W\| < \|T^{-1}\|\,\|W\| < 1$.

Good condition $\|W\| < \frac{1}{\|T^{-1}\|}$

$\cancel{\|T^{-1} \circ T\| \leq \|T^{-1}\|\,\|T\| \Rightarrow \frac{1}{\|T^{-1}\|} \leq \|T\|}$