B1 <u>Hamel basis</u> = finitely linearly independent set $\{e_\alpha\}_{\alpha\in A}$ such that for any $x\in X$ there is unique $\cancel{X_i}\in\mathbb{R}$ $x=\sum_{i=1}^{n} a_i e_{\alpha_i}$.

<u>Schauder basis</u> = countable ~~linearly~~ linearly independent set $\{e_i\}$ such that for any $x\in X$ there are unique $\cancel{X_i}\in\mathbb{R}$ $x=\sum_{i=1}^{\infty} x_i e_i$ (convergence in norm $\Rightarrow$ requires topology).

<u>Fact:</u> Any Banach space has uncountable Hamel basis.

<u>Proof:</u> Suppose it has countable basis. Write $X_n=\{x\in X : \mathrm{span}\{x_1,\dots,x_n\}\}$. By assumption, $\bigcup X_n = X$ and since X_n is finite dimensional, ~~By~~ By BCT, there is n s.t. X_n does not have empty interior.

So $\exists_{B(x,r)\subset X_n}$. Let $z\in X$, take $y=x+r\frac{z}{\|z\|}\in X_n$

but X_n is a linear subspace so $z=\frac{\|z\|}{r}\left(y-\underset{X_n}{\underset{\cap}{x}}\right)\in X_n$

So that $X_n=X$, contradiction.

$\square$.

B2 These spaces have countable Hamel basis so the assertion follows by B1.

B5 The same type of proof as B1: $X_n=\{x\in X : \sup_{\alpha\in A}\|T_\alpha x\|\le n\}$.

$\Rightarrow$ by BCT $\exists_n$ X_n has nonempty interior $\longrightarrow$ put the ball inside....

(U4) Operators should be indexed with $\varphi \in \mathcal{B}$. They act on $x \in X$.

$$T_\varphi(x) = \varphi(x) \qquad T: X \to \mathbb{R}, \quad \varphi \in A$$

For every $x \in X$ $\sup_{\varphi \in A} |T_\varphi(x)| < \infty \Rightarrow \sup_{\|x\| \leq 1} \sup_{\varphi \in A} |T_\varphi(x)| < \infty$

$$\Rightarrow \sup_{\varphi \in A} \|\varphi\| < \infty.$$

(U5) Small homework $\rightsquigarrow$ next week.

$y \mapsto a(x,y)$ is continuous for each fixed $x \in E$

$\Leftrightarrow \quad |a(x,y)| \leq C_x \|y\|$

Similarly, $|a(x,y)| \leq C_y \|x\|$ $\qquad \Rightarrow \exists_{C} \; |a(x,y)| \leq C \|x\| \|y\|$.

(C independent of x and y).

U2 • For any $x \in X$, $T_n x$ converges, hence it is bounded. UBP implies $\sup_n \|T_n\| < \infty$.

• T is linear: $T(x+y) = \lim_{n\to\infty} T_n(x+y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty$.

• $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$: $\quad T_n(x) \leq \|T_n\| \, \|x\|$

$\Rightarrow$

$$\|T_n x\|_Y = \liminf_{n\to\infty} \|T_n x\|_Y \leq \left(\liminf_{n\to\infty} \|T_n\|\right) \|x\| \Rightarrow$$

$$\|T\| \leq \liminf \|T_n\|$$

• $x_n \to x$ in X: $$\|T_n x_n - Tx\|_Y \leq \underbrace{\|T_n x_n - T_n x\|_Y}_{\leq \left(\sup_n \|T_n\|\right) \|x_n - x\|_X} + \underbrace{\|T_n x - Tx\|_Y}_{\to 0 \text{ as } n \to \infty \text{ by pointwise convergence}}$$

Hence, $\|T_n x_n - Tx\|_Y \to 0$ as $n \to \infty$.

U3 Let T_n be projection in ℓ^1 on n-th coordinate. Since $\sum |x_i|$ is convergent, sequence $x_i \to 0$ and hence, for any $x \in \ell^1$, $T_n x \to 0$.

But $T_n \cancel{x} \longrightarrow 0$ in $\mathcal{L}(\ell^1, \mathbb{R})$. Indeed,

$$\limsup_{n\to\infty} \sup_{\|x\|_{\ell^1} \leq 1} |T_n(x)| \underset{x = e_n}{\geq} \lim_{n\to\infty} |T_n(e_n)| = 1.$$

(U6) If f is fixed, $|B(f,g)| \leq \|f\|_\infty \int |g(t)|\, dt \leq$

$$\leq \|f\|_\infty \|g\|_1 =: C_f \|g\|_1$$

So indeed B is separately continuous. Suppose it is jointly continuous. There is a constant C s.t.

$$|B(f,g)| \leq C\left(\int |f|\right)\left(\int |g|\right)$$

In particular, for $f=g$ we have $\int_0^1 f^2 \leq C\left(\int_0^1 |f|\right)^2$.

$f = t^n$

$$\int_0^1 f^2 = \int_0^1 t^{2n}\, dt = \frac{1}{2n+1}$$

$$\int |f|\, dt = \int_0^1 t^n\, dt = \frac{1}{n+1} \quad \Rightarrow \frac{1}{2n+1} \leq C \frac{1}{(n+1)^2}$$

$\Rightarrow$ contradiction.

(U7) $\rightsquigarrow$ Big Homework

$\square$.