Functional Analysis (WS 19/20), Problem Set 4

(Open Mapping Theorem and Closed Graph Theorem)

Open Mapping Theorem Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator that is surjective. Then, T is open i.e. there is a constant $c > 0$ such that

$$B_Y(0, c) \subset T(B_X(0, 1)).$$

Inverse Mapping Theorem Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator that is bijective. Then, T^{-1} is also bounded.

Closed Graph Theorem Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ a linear operator. Then, T is bounded if and only if its graph

$$G(T) = \{(x, Tx) : x \in X\}$$

is closed in the product space $X \times Y$.

Open Mapping Theorem

O1. Let $X = (l_1, \|\cdot\|_1)$ and $Y = (l_1, \|\cdot\|_\infty)$. Prove that identity operator $T : X \to Y$ defined with $Tx = x$ is a bounded linear map that is not open. Why OMT does not hold in this case?

O2. (**Inverse Mapping Theorem**) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator that is bijective. Then, T^{-1} is also bounded.

O3. Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X and suppose that X is a Banach space with respect to both of them. Finally, suppose that $\|x\|_1 \leq C\|x\|_2$ for some constant C. Then, there is a constant c such that

$$\|x\|_2 \leq c\|x\|_1$$

and hence, both norms are equivalent on X.

O4. Prove that $C[0, 1]$ equipped with $L^p(0, 1)$ norm is not a Banach space for $1 \leq p < \infty$. *Remark:* Compare this problem with exercises like "set l^1 with l^∞ norm is not a Banach space".

O5. Let $(X, \|\cdot\|_X)$ be a normed space. Suppose that $A, B \subset X$. Prove that $\overline{A} + \overline{B} \subset \overline{A + B}$ where "+" stands for the Minkowski sum of sets.

Closed Graph Theorem

C1. (**Closed Graph Theorem**) Use Problem O3. to deduce Closed Graph Theorem. *Hint:* For $x \in X$, consider two norms: (a) $\|x\|_X$ and (b) $\|x\|_X + \|Tx\|_Y$.

C2. (Uniform Boundedness Principle again) Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces. Suppose that $T_n : X \to Y$ are bounded linear operators such that $T_n \to T$ pointwisely (i.e. for every $x \in X$, $T_n x \to Tx$ in Y). Prove that if $x_n \to x$ in X, then $T_n x_n \to Tx$ in Y. *Note:* This is Problem U2 from Problem Set 3 again.

C3. Show that, up to an equivalence of norms, the supremum norm is the only norm on $C[0, 1]$ which makes $C[0, 1]$ complete and which also implies the pointwise convergence.

C4. Show that, up to an equivalence of norms, the $\|\cdot\|_p$ norm is the only norm on $L^p(0,1)$ which makes $L^p(0,1)$ complete and which also implies pointwise converges a.e. of some subsequence.

C5. Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x \in X$:

$$(Tx)(x) \geq 0.$$

Prove that T is a bounded operator. *Clarification:* For any $x \in X$, $Tx \in X^*$ so $(Tx)(x)$ is just a real number i.e. functional Tx evaluated at x. *Hint:* If one has to show that $\varphi_1, \varphi_2 \in X^*$ satisfy $\varphi_1 = \varphi_2$, it is convenient to fix arbitrary $x \in X$ and prove $\varphi_1(x) = \varphi_2(x)$.

C6. Let $(X, \|\cdot\|_X)$ be a Banach space. Consider a linear operator $T : X \to X^*$ such that for all $x, y \in X$:

$$(Tx)(y) = (Ty)(x)$$

(see clarification in Problem C5. if necessary). Prove that T is a bounded operator. *Hint:* Let $x_n \to x$ and consider $(Tx_n)(x)$.

Introduction to invertibility - norm condition[1]

I1. (**necessary condition for series convergence**) Let $(X, \|\cdot\|_X)$ be a normed space. Suppose that $\sum_{k=0}^{\infty} x_k$ converges in $(X, \|\cdot\|_X)$. Prove that $x_k \to 0$.

I2. Let $(X, \|\cdot\|_X)$ be a normed space and $T \in \mathcal{L}(X, X)$. Prove that if $\sum_{k=1}^{\infty} T^k$ converges in $\mathcal{L}(X, X)$ then $(I - T)^{-1}$ exists and

$$(I - T)^{-1} = \sum_{k=1}^{\infty} T^k.$$

Moreover, if $(X, \|\cdot\|_X)$ is a Banach space, it is sufficient that $\sum_{k=1}^{\infty} \|T\|^k < \infty$, i.e. $\|T\| < 1$.

I3. Let $k \in C([0,1] \times [0,1])$ with $\|k\|_\infty < 1$ and $y \in C([0,1])$. Prove that there is a unique continuous solution $x \in C([0,1])$ to the integral equation

$$x(t) - \int_0^1 k(s,t)x(s)\, ds = y(t).$$

I4. Let $(X, \|\cdot\|_X)$ be a Banach space. Prove that the set of invertible operators is open in $\mathcal{L}(X, X)$ equipped with operator norm. *Hint:* Consider ball in $\mathcal{L}(X, X)$ centered at T. If S is in that ball, write $S = T + W = T(I + T^{-1}W)$ for some "small" W. **Note**: Any bounded linear operator that is invertible can be perturbed (in a sufficiently small way) and the resulting perturbation is still invertible.

[1] This is content of Problem Set 2. It is copied here for your convenience.