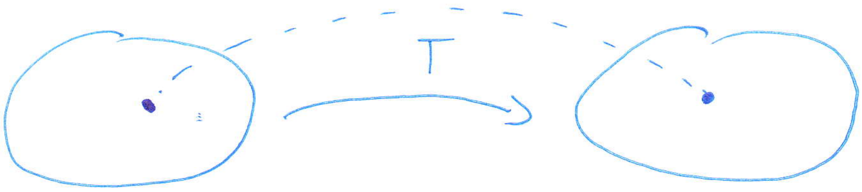
Problem Set 4

(Open Mapping Theorem + Closed Graph Theorem)

Q1 Clearly $\|T\| = 1$, T is linear, bounded etc. Suppose it is open.



$X = \ell^1, \|\cdot\|_1$ $\qquad\qquad$ $\ell^1, \|\cdot\|_\infty = Y$

Then, $\exists_c\, T(B_X(0,1)) \supset B_Y(0,c) \Rightarrow \exists_c\, B_X(0,1) \supset B_Y(0,c)$

$\Rightarrow \forall_{x \in \ell^1} \ \|x\|_\infty < c \Rightarrow \|x\|_1 < 1 \quad (*)$

Let n be the smallest integer s.t. $n \cdot c > 1$. We take

$x = (\underbrace{c, c, c, \dots, c}_{n \text{ times}}, 0, 0, \dots) \in \ell^1$ and contradicts $(*)$.

Q2 $\exists_c\, T(B_X(0,1)) \supset B_Y(0,c) \xRightarrow{\text{injectivity!}} \|Tx\|_Y < c \Rightarrow \|x\|_X < 1.$

T bijective $\Rightarrow T^{-1}$ exists

Claim: $\|x\|_X \leq \frac{1}{c}\|Tx\|_Y$. Indeed, take $\tilde{x}_\varepsilon = \frac{x}{\|Tx\|} c(1-\varepsilon)$

Then $\|T\tilde{x}_\varepsilon\|_Y = c(1-\varepsilon) < c$ so $\|\tilde{x}_\varepsilon\|_X < 1$

$$\|\tilde{x}_\varepsilon\|_X = \frac{\|x\|_X}{\|Tx\|_Y} c(1-\varepsilon) < 1$$

$\|x\|_X \leq \frac{1}{c}\|Tx\|_Y$ send $\varepsilon \to 0$ $\Leftarrow$ $\frac{\|x\|_X}{\|Tx\|_Y} c(1-\varepsilon) < 1$ $\square$.

(03) Consider identity $T : (X, \|\cdot\|_2) \longrightarrow (X, \|\cdot\|_1)$, $T = id$.

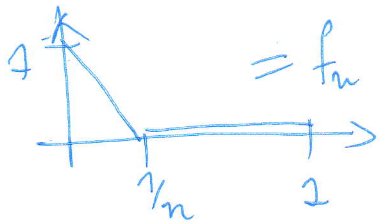
T is bounded, bijective, linear. $\Rightarrow$ T^{-1} is also bounded, hence

$\|x\|_2 \le c\,\|x\|_1$ for some constant $c > 0$. by (02) $\square$.

(04) Suppose it is a Banach space $(C([0,1]), \|\cdot\|_p)$. We know that $(C([0,1]), \|\cdot\|_\infty)$ is Banach. Moreover, $\|f\|_p \le C_p \|f\|_\infty$. $\Rightarrow$

$\|f\|_\infty \le C\,\|f\|_p$. $\forall f \in C([0,1])$.

Contradiction with



$= f_n$

$\|f_n\|_\infty = 1$

$\|f_n\|_p \longrightarrow 0$

as $n \to \infty$. $\checkmark$

(05) $A + B = \{a + b : a \in A, b \in B\}$.

Claim: $\overline{A} + \overline{B} \subset \overline{A+B}$

Let $x \in \overline{A} + \overline{B}$. Then $x = a + b$ where $a \in \overline{A}$, $b \in \overline{B}$

$\Downarrow$ $\exists\, a_n \in A$, $a_n \to a$

$\Downarrow$ $\exists\, b_n \in B$, $b_n \to b$

$\Rightarrow$ $a_n + b_n \in A + B$, $a_n + b_n \to a + b$ $\Rightarrow$ $a + b \in \overline{A+B}$.

$\square$.

(C1) Clearly $\|x\|_X \le \|x\|_X + \|Tx\|_Y$ and $(X, \|\cdot\|_X)$ is Banach.

We claim that $(X, \|x\|_X + \|Tx\|_Y)$ is also Banach space

Let x_n be Cauchy $\Rightarrow$ $\{x_n\}$ Cauchy in X $\Rightarrow$ $x_n \to x$ in X

$\{Tx_n\}$ Cauchy in Y $\Rightarrow$ $Tx_n \to y$ in Y

~~But~~ so $(x_n, Tx_n) \to (x, y)$ in $X \times Y$. But $(x_n, Tx_n) \in G(T)$ and $G(T)$ is closed $\Rightarrow$ $y = Tx$. (as $(x,y) \in G(T)$). $\square$.

(If T is bounded linear operator and $x_n \to x$ then $Tx_n \to Tx$ so the graph $G(T)$ is closed).

(C2) Once again,

$$\|T_n x_n - Tx\|_Y \le \underbrace{\|T_n x_n - T_n x\|}_{\le (\sup_n \|T_n\|) \|x_n - x\|} + \underbrace{\|T_n x - Tx\|}_{\to 0 \text{ by pointwise convergence}}$$

(C3) Suppose there is another norm on $C[0,1]$ which makes it Banach and implies ptwise convergence.

We write $X = (C[0,1], \|\cdot\|_\infty)$ – standard $C[0,1]$ space

$Y = (C[0,1], \|\cdot\|_A)$ – $C[0,1]$ with $\|\cdot\|_A$ norm we study

By assumption, X, Y are Banach spaces. Consider $T: X \to Y$, $T = Id$.

We study closedness of graph of T. $G(T) = \{(f, f) \in X \times Y\}$.

Suppose $f_n \to f$ in X. We need $f = g$.

$f_n \to g$ in Y.

Since $f_n \xrightarrow[(\text{in } X)]{} f \Rightarrow f_n \to f$ pointwise

Since $f_n \to g$ in $Y \Rightarrow f_n \to g$ pointwise (as we <u>assumed</u> that convergence in $\|\cdot\|_A$ implies ptwise convergence). Hence, $f = g$.

$\Rightarrow$ T is bounded by CGT and $\|f\|_A \leq C\|f\|_\infty$

Similar argument shows $\|f\|_\infty \leq \tilde{C}\|f\|_A$. □.

C4 Exactly the same like C3 but this time convergence of subsequence is used.

□.

C6 $T : X \to X^*$. We study graph of T and claim it is closed.

$G(T) = \{(x, Tx) \subset X \times X^*\}$.

Let $(x_n, Tx_n) \in G(T)$ and $(x_n, Tx_n) \longrightarrow (x, y)$ in $X \times Y$.

We need to prove $y = Tx$.

As in the Hint, consider

$$
\begin{tikzcd}
(Tx_n)(z) \arrow[r, equal, "\text{ASSUMPTION}"] \arrow[d] & (Tz)(x_n) \arrow[d] \\
y(z) & (Tz)(x) \arrow[d, equal] \\
\underline{y = Tx.} & (Tx)(z) \arrow[l, Rightarrow]
\end{tikzcd}
$$

C5 $\rightsquigarrow$ small homework for next week

I1 In any metric space, convergent sequences are Cauchy.

Hence, $\forall_{\varepsilon>0} \exists_N \forall_{n,m\geq N} \|S_n - S_m\| \leq \varepsilon$ where $S_n = \sum_{i=1}^{n} x_i$.

Take $m = n+1 \Rightarrow \forall_{\varepsilon>0} \exists_N \forall_{n\geq N} \|x_n\| \leq \varepsilon \Rightarrow x_n \to 0$.

I2 We have to check bijectivity i.e. existence of left and right inverses.

- If $f: X \to X$ has left inverse $g: X \to X$ (i.e. $g(f(x)) = x$) then f is injective (as $f(x) = f(y) \Rightarrow x = y$).
- If $f: X \to X$ has right inverse $g: X \to X$ (i.e. $f(g(x)) = x$) then f is surjective (as $\exists_x \forall_y f(y) \neq x \underset{\uparrow y=g(x)}{\Rightarrow}$ contradiction).

We prove that $\sum_{k\geq 0} T^k$ is left and right inverse of $(I-T)$.

Let $S_k = \sum_{k=0}^{k} T^k$. Note that $S_k(I-T) = (I-T)S_k$. Therefore, for any $x \in X$:

$$x = Ix = \lim_{m\to\infty}(I - T^{m+1})x \overset{T^n \to 0}{=} \lim_{m\to\infty}(I - T + T - T^2 + T^2 - T^3 + \dots - T^{m+1})x$$

$$= \lim_{m\to\infty}(I-T)S_m x \underset{\uparrow \text{continuity}}{=} (I-T)\lim_{m\to\infty} S_m x \underset{\uparrow \text{convergence of } \sum T^n}{=} (I-T)\left(\sum_{n=0}^{\infty} T^n\right)x.$$

Similarly, $\left(\sum_{n=0}^{\infty} T^n\right)(I-T)x = x$.

Hence $\sum_{n=0}^{\infty} T^n = (I-T)^{-1}$.

In case of BS: $(I-T)$ is invertible if $\|T\| < 1$. Then,

$$\|(I-T)^{-1}\| \leq \frac{1}{1-\|T\|}$$

□.

(I3) We study operator $T: C[0,1] \to C[0,1]$ given with

$$(Tx)(t) = \int_0^1 k(s,t)\, x(s)\, ds \Rightarrow \|T\| \leq \|k\| < 1.$$

Hence $I - T$ is invertible $\Rightarrow x = (I-T)^{-1} y$ and has to be unique.

(I4) Let $T \in \mathcal{L}(X,X)$. We want to find a ball in $\mathcal{L}(X,X)$ centered in T s.t. all operators in that ball are invertible (so we need to find radius of such ball). If S is in that ball we have

$$S = T + W = T(I + T^{-1}W) = T(I - (-T^{-1}W))$$

and this is invertible provided $\|T^{-1}W\| < 1$.

We estimate $\|T^{-1}W\| < \|T^{-1}\| \|W\| < 1$.

Good condition $\|W\| < \frac{1}{\|T^{-1}\|}$

✓.