Functional Analysis (WS 19/20), Problem Set 5

(Introduction to Hilbert Spaces)

Basic properties of Hilbert spaces

P1. Verify that $L^2(0,1)$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$ is a Hilbert space.

P2. Verify that the space

$$X = \left\{ f \text{ measurable and } \int_0^1 |f(t)|^2 e^t dt < \infty \right\}$$

is a Hilbert space with the inner product $\langle f, g\rangle = \int_0^1 f(x)\, g(x) e^t\, dx$. Check that it is a subset of $L^2(0,1)$.

P3. More generally, let $w \in L^1(0,1)$ be a nonnegative function. Verify that the space

$$X = \left\{ f \text{ measurable and } \int_0^1 |f(t)|^2 w(t) dt < \infty \right\}$$

is a Hilbert space with the inner product $\langle f, g\rangle = \int_0^1 f(x)\, g(x) w(t)\, dx$. Is it a subset of $L^2(0,1)$?

P4. Verify that $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$ defines an inner product on $C[0,1]$. Is $(C[0,1], \langle \cdot, \cdot \rangle)$ a Hilbert space?

P5. We say that a Banach space $(X, \|\cdot\|)$ is uniformly convex if for any $\epsilon > 0$, there is $\delta > 0$ such that for any $x, y \in E$:

$$\text{if } \|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \epsilon \text{ then } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

Prove that any Hilbert space is uniformly convex.[1]

P6. (**Pythagorean Theorem**) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $x, y \in H$. Prove that

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

P7. (**Bessel's inequality**) Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal sequence in Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Prove that for any $x \in H$

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

P8. For $1 \leq p \leq \infty$ consider space $L^p(0,1)$. Prove that norm $\|\cdot\|_p$ satisfies parallelogram identity:

$$2\, \|x\|_p^2 + 2\, \|y\|_p^2 = \|x + y\|_p^2 + \|x - y\|_p^2$$

if and only if $p = 2$. *Hint:* Consider functions with disjoint supports.

[1] A deep result due to Milman and Pettis asserts than any uniformly convex Banach space E is reflexive, i.e. $E^{**} = E$ up to an isometric isomorphism. This somehow connects *geometric* and *analytical* properties of Banach spaces. Note that this is still weaker than Riesz Representation Theorem asserting that for any Hilber space H we have $H = H^*$.

Orthogonal complements and projections

O1. Let $K \subset H$ be non-empty, convex and closed subset of Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. We know that for every $f \in H$, there exists a unique $P_K(f) \in K$ such that

$$\inf_{v \in K} \|f - x\| = \|f - P_K(f)\|$$

and $P_K(f)$ is usually called a projection of f on K. Prove that $P_K(f)$ can be equivalently characterized as

$$P_K(f) \in K \text{ and } \langle v - P_K(f), f - P_K(f) \rangle \leq 0 \text{ for all } v \in K.$$

O2. For a subset $K \subset H$ we define its orthogonal complement

$$K^\perp = \{x \in H : \langle x, v \rangle = 0 \text{ for all } v \in K\}.$$

Prove that $K^\perp$ is a closed linear subspace of H.

O3. Prove that the projection $P_K(f)$ is 1-Lipschitz.

O4. Let $M \subset H$ be a linear closed subspace of Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. Prove that projection $P_M : H \to M$ defines a bounded linear operator. Moreover, for any $f \in H$, $P_M(f)$ can be equivalently characterized as

$$P_M(f) \in M \text{ and } \langle v, f - P_M(f) \rangle = 0 \text{ for all } v \in M.$$

Note that $f - P_M(f) \in M^\perp$.

O5. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $M \subset H$ a closed subspace. Prove that $H = M \oplus M^\perp$, i.e. for any $x \in H$ we can write $x = u + v$ where $u \in M$ and $v \in M^\perp$ are uniquely determined. Moreover, there is a bounded linear map $P : H \to H$ with range M such that $P(x) = u$.

O6. Consider

$$X = \{f \in L^2(0,1) : f(x) = 0 \text{ for all } x \in [0, 1/2]\}$$

as a subspace of $L^2(0,1)$. Find $X^\perp$ and compute projection operator on X.

O7. Consider

$$X = \{f \in L^2(-1,1) : f(x) = f(-x)\}$$

as a subspace of $L^2(-1,1)$. Find $X^\perp$ and compute projection operator on X.

O8. Find a polynomial $w(t)$ of degree at most 2 such that $\int_0^1 |w(t) - t^4|^2 dt$ is the smallest.

O9. Find a polynomial $w(t)$ of degree at most 1 such that $\int_0^1 |w(t) - \sqrt{t}|^2 dt$ is the smallest.

Riesz Representation Theorem

R1. Prove that there exists a function $f \in L^2(0,1)$ such that $\int_0^1 t^2 f(t)dt = \int_0^1 e^t f(t)dt$. Is this function uniquely determined?

R2. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a symmetric bilinear continuous form that is coercive, i.e. there is a constant β such that $a(u,u) \geq \beta\|u\|^2$. Prove that $(H, a(\cdot,\cdot))$ is a Hilbert space with the same topology as $(H, \langle\cdot,\cdot\rangle_H)$ (i.e. norms are equivalent).

Hint: It may be helpful to recall the proof of the Cauchy-Schwartz inequality.

R3. **(Lax-Milgram Lemma, simplified version)** Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a symmetric bilinear continuous form that is coercive, i.e. there is a constant β such that $a(u,u) \geq \beta\|u\|^2$. Let $l \in H^*$. Prove that there is a unique $u \in H$ such that

$$a(u,v) = l(v) \text{ for all } v \in H.$$

R4. Prove that there is a uniquely determined function $u \in L^2(0,1)$ such that for all $v \in L^2(0,1)$

$$\int_0^1 u(t)v(t)e^t dt = \int_0^1 \sin(t)v(t)dt.$$

Complement of c_0 in l^∞ does not exist

In the following set of exercises, we prove so-called Phillips Lemma asserting that c_0 cannot be complemented as a closed subset of l^∞. We follow the approach from mathematical StackExchange post "Complement of c_0 in l^∞" [2] by **t.b.**.

C1. Let S be a countable infinite set. Prove that there is an uncountable almost disjoint family of infinite subsets of S, i.e. there is a family $\{A_i\}_{i\in I}$ of subsets of S such that $|I| = |\mathbb{R}|$, $|A_i| = |\mathbb{N}|$ and $A_i \cap A_j$ is finite whenever $i \neq j$. *Hint:* Think of $S = \mathbb{Q} \cap [0,1]$ and $I = [0,1] \setminus S$.

C2. Let $P : l^\infty \to l^\infty$ be a bounded linear operator such that $P(x) = 0$ for all $x \in c_0$. Prove that there exists an infinite set of $\mathbb{N}$ such that $P(x) = 0$ for all x supported on A.

C3. Prove that c_0 is not complemented in l^∞.

[2] https://math.stackexchange.com/questions/132520/complement-of-c-0-in-ell-infty

Orthogonal bases in Hilbert spaces

B1. Let $(H, \langle \cdot, \cdot \rangle_H)$ be an infinite dimensional Hilbert space. Prove that H has an orthonormal basis (in the sense of Schauder) if and only if H is separable. Moreover, if $\{e_i\}_{i \in I}$ is an orthonormal basis of H, we have

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$$

and this series converges in H.

B2. Let $(H, \langle \cdot, \cdot \rangle_H)$ be an infinite dimensional separable Hilbert space. Prove that H is isometrically isomorphic to l^2 with its standard inner product $\langle \cdot, \cdot \rangle_{l^2}$, i.e. there is an isomorphism $T : H \to l^2$ such that $\langle Tu, Tv \rangle_{l^2} = \langle u, v \rangle_H$. *Hint*: Use Problem B1.

B3. (**Parseval's identity**) Let $\{e_i\}_{i \in I}$ be an orthonormal basis of Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. Prove that

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2.$$

B4. There are Hilbert spaces that are not separable. Consider space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ only for countably many $x \in \mathbb{R}$ equipped with the scalar product

$$\langle f, g \rangle = \sum_{x \in \mathbb{R}} f(x)\, g(x).$$

Prove that it is a Hilbert space but it is not separable (find uncountable set of elements x_i such that $\|x_i - x_j\| \geq 1$).

B5. Prove that sequence of polynomials

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$$

defines orthogonal basis of $L^2(-1, 1)$ (definitely, we won't do that exercise in class).