Problem Set 5
(Introduction to Hilbert Spaces)

Hilbert Space = Banach space + inner product $\langle \cdot , \cdot \rangle : H \times H \to \mathbb{R}$ such that:

(i) $\langle \cdot , \cdot \rangle$ is bilinear

(ii) $\langle x, x \rangle = \|x\|^2$

(iii) $\langle x, y \rangle = \langle y, x \rangle$

P1 Obvious

P2 We note that $X = \{ f \text{ measurable} : \int_0^1 |f|^2 e^t \} \subset L^2(0,1)$ as $\int_0^1 |f|^2 e^t dt \geq \int_0^1 |f|^2 dt$ as $e^t \geq 1 = e^0$. We only need to check it is closed in $L^2(0,1)$. Let $f_n \in X$, $f_n \to f$ in L^2. We need to prove that $\int_0^1 |f|^2 e^t dt < \infty$.

But $\int_0^1 |f|^2 e^t dt \leq e^1 \int_0^1 |f|^2 dt = e^1 \lim_{n\to\infty} \int_0^1 |f_n|^2 dt \leq$ ~~bounded~~ bounded as convergent sequences are bounded.

P4 Well, we checked at least 1000 times that $(C[0,1], \|\cdot\|_2)$ is not a Banach space (last time, this was conclusion from Inverse Mapping Theorem).

P5 $\forall_{x,y \in H}$ $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \iff$

$$\frac{1}{2}(\|x\|^2 + \|y\|^2) = \left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2$$

Let x, y be s.t. $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$. Then,

$$\left\| \frac{x+y}{2} \right\|^2 \leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) - \left\| \frac{x-y}{2} \right\|^2 \leq$$

$$\leq 1 - \left\| \frac{x-y}{2} \right\|^2 \leq 1 - \frac{\varepsilon^2}{4} = (1-\delta)^2$$

$\Rightarrow$ we choose $\delta = 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2}$.

Mention here Millman-Pettis Theorem: if E unif. convex $\Rightarrow E^{**} = E$ (geometry $\Rightarrow$ analysis)

Note that $\ell^1(\mathbb{R}^2)$, $\ell^\infty(\mathbb{R}^2)$

P3 We define measure on $[0,1]$ with $\mu(A) = \int_A w(x)\, d\lambda(x)$, it is a finite measure. Hence, $L^2([0,1], \mu)$ is a Banach space, i.e. space of measurable f s.t.

$$\int_0^1 |f|^2 d\mu = \int_0^1 |f|^2 w(x)\, dx$$

so it is precisely X. Inner product is given by $\int_0^1 f(x) g(x) w(x)\, dx$.

P7 First, consider finite sum. We write

$$0 \leq \left\| x - \sum_{i=1}^{n} (x, e_i) e_i \right\|^2 = \|x\|^2 + \left\| \sum_{i=1}^{n} (x, e_i) e_i \right\|^2$$

$$- 2\left(x, \sum_{i=1}^{n} (x, e_i) e_i\right)$$

$$\left\| \sum_{i=1}^{n} (x,e_i)e_i \right\|^2 = \left(\sum_{i=1}^{n}(x,e_i)e_i, \sum_{i=1}^{n}(x,e_i)e_i \right) \underset{\substack{\uparrow \\ \text{orthogonality}}}{=}$$

$$= \sum (x,e_i)^2 \|e_i\|^2 \underset{\substack{\uparrow \\ \text{orthonormality}}}{=} \sum (x,e_i)^2$$

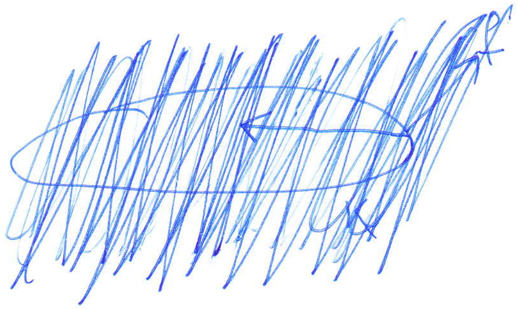
$$\left(x, \sum_{i=1}^{n}(x,e_i)e_i\right) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \sum_{i=1}^{n}(x,e_i)(x,e_i) = \sum_{i=1}^{n}(x,e_i)^2$$

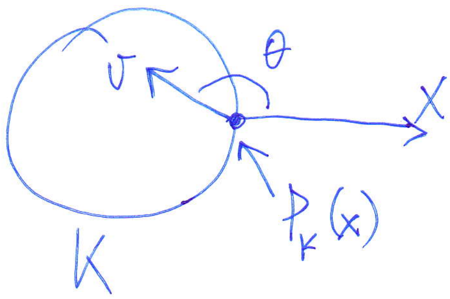
$\Rightarrow \displaystyle\sum_{i=1}^{n}(x,e_i)^2 \leq \|x\|^2$. Now, let $n \to \infty$. $\square$.

P6 $\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + \|y\|^2 + 2(x,y)$. $\square$.

P8 $\rightsquigarrow$ small homework

O1





(1) $\forall_v \quad (v - P_K(x), x - P_K(x)) \leq 0$ (equivalent characterization)

(2) $\|x - P_K(x)\| = \inf_{v \in K} \|x - v\|$

$(\forall v \in K)$

(2) $\Rightarrow$ (1): $\|x - P_K(x)\|^2 \leq \|x - v\|^2$. Choose $v = t P_K(x) + (1-t) w \in K$ if $w \in K$.

$$\|x-(tP_K(x)+(1-t)w)\| = \|x-P_K(x)-(1-t)(w-P_K(x))\|$$

$$\Rightarrow \|x-P_K(x)\|^2 \le \|(x-P_K(x))-(1-t)(w-P_K(x))\|^2 =$$

$$= \|x-P_K(x)\|^2 + (1-t)^2\|w-P_K(x)\|^2 - 2(1-t)\left(x-P_K(x), w-P_K(x)\right)$$

$$\Rightarrow (1-t)\|w-P_K(x)\|^2 \ge 2\left(x-P_K(x), w-P_K(x)\right)$$

send $t \to 1^- \Rightarrow \left(x-P_K(x), w-P_K(x)\right) \le 0$.

$(1) \Rightarrow (2)$: $\forall_{v\in K}$ We want to check:

$$\|x-P_K(x)\|^2 - \|\underbrace{x-v}_{x-P_K(x)+P_K(x)-v}\|^2 \le 0 \quad \forall_{v\in K}$$

$$\|x-v\|^2 = \|x-P_K(x)\|^2 + \|P_K(x)-v\|^2 + 2\left(x-P_K(x), P_K(x)-v\right)$$

$$\Rightarrow \|x-P_K(x)\|^2 - \|x-v\|^2 = \underbrace{-\|P_K(x)-v\|^2}_{\le 0} - 2\underbrace{\left(x-P_K(x), P_K(x)-v\right)}_{\ge 0} \le 0$$

✓.

Q2 $K^\perp = \{x \in H : (x,v)=0 \ \forall_{v\in K}\}$.

$K^\perp$ is linear space as if $x,y \in K \Rightarrow (x+y,v)=0 \ \forall_{v\in K}$.
$K^\perp$ is closed: let $x_n \in K^\perp$ i.e. $(x_n,v)=0 \ \forall_{v\in K}$ and $x_n \to x$ in H. Then

$$(x,v) = \underbrace{(x_n,v)}_{=0} + \underbrace{(x-x_n,v)}_{\to 0 \text{ as } n\to\infty}.$$

✓.

(03) Let $f_1, f_2 \in H$. We claim $\| P_K(f_1) - P_K(f_2) \| \leq \| f_1 - f_2 \|$

We know $\forall_{v \in K}$:

$$(v - P_K(f_1), f_1 - P_K(f_1)) \leq 0 \quad \forall_{v \in K} \qquad v := P_K(f_2)$$
$$(v - P_K(f_2), f_2 - P_K(f_2)) \leq 0 \quad \forall_{v \in K} \qquad v : P_K(f_1).$$

$$(P_K(f_2) - P_K(f_1), f_1 - P_K(f_1)) + (\underbrace{P_K(f_1) - P_K(f_2)}_{= -(P_K(f_2) - P_K(f_1))}, f_2 - P_K(f_2)) \leq 0$$

$$\Rightarrow \| P_K(f_2) - P_K(f_1) \|^2 \leq (P_K(f_2) - P_K(f_1), f_2 - f_1) \underset{\text{C-S}}{\leq}$$

$$\leq \| P_K(f_2) - P_K(f_1) \| \, \| f_2 - f_1 \| \Rightarrow \checkmark.$$

(04) P_M is 1-Lipschitz so it is bounded linear operator

(we need M to be subspace so that operator is well-defined).

(*) Proof of characterization: $P_M(f) \in M$ and $(v, f - P_M(f)) = 0 \ \forall_{v \in M}$.

(*) $\Rightarrow$ projection   We need to check $\forall_{v \in M} \ (v - P_M(f), f - P_M(f)) \leq 0$

But $(v - P_M(f), f - P_M(f)) = \underbrace{(v, f - P_M(f))}_{=0} - \underbrace{(P_M(f), f - P_M(f))}_{=0} = 0 \leq 0$ as $P_M(f) \in M$.

projection $\Rightarrow$ (*) $(v - P_M(f), f - P_M(f)) \leq 0 \quad \forall_{v \in M}$

$$\Rightarrow \forall_{t \in \mathbb{R}} \ \forall_{v \in M} \quad \underbrace{(tv - P_M(f), f - P_M(f))}_{} \leq 0$$

$$= t(v, f - P_M(f)) \leq (P_M(f), f - P_M(f))$$

$(v, f - P_M(f)) \leq 0$ $\quad$ ($t \to \infty$). $\forall_{v \in M}$

Take $-v \in M$ as M is linear space $\Rightarrow$ $(v, f - P_M(f)) = 0$.

05 By 04 $\quad f = \underbrace{P_M(f)}_{\in M} + \underbrace{f - P_M(f)}_{\in M^\perp}$. This decomposition is unique. ~~Indeed: $M \cap M^\perp = \{0\}$~~. Indeed, suppose $f = \underset{M}{\underset{\cap}{x_1}} + \underset{M^\perp}{\underset{\cap}{y_1}}$

and $f = \underset{M}{\underset{\cap}{x_2}} + \underset{M^\perp}{\underset{\cap}{y_2}}$ $\Rightarrow$ $0 = \underset{M}{\underset{\cap}{(x_1 - x_2)}} + \underset{M^\perp}{\underset{\cap}{(y_1 - y_2)}}$

$\Rightarrow$ $x_1 - x_2 \in M$ and $x_1 - x_2 \in M^\perp$ ($M, M^\perp$ is linear space...)

Suppose $x \in M$ and $x \in M^\perp$, i.e. $\forall_{v \in M} \langle x, v \rangle = 0$. Choose $v = x \Rightarrow \underline{x = 0}$.

$\square$.

06 $\quad X = \{ f \in L^2(0,1) : f|_{[0, \frac{1}{2}]} = 0 \}$.

$$X^\perp \overset{\text{def}}{=} \{ f \in L^2(0,1) : \int f(x)\, g(x) = 0 \quad \forall_{g \in L^2(0,1),\ g|_{[0,\frac{1}{2}]} = 0} \}$$

Claim $\quad X^\perp = \{ f \in L^2(0,1) : f|_{[\frac{1}{2}, 1]} = 0 \}$.

<u>Proof</u>: $\supseteq$ is trivial. Let $f \in X^\perp$, i.e. $\forall_{g \in L^2,\ g|_{[0,\frac{1}{2}]} = 0} \int_0^1 fg = 0$.

Suppose that $f > 0$ on some set of positive measure in $[\frac{1}{2}, 1]$.

In part, there is k s.t. $f > \frac{1}{k}$ on some set of positive measure in $[\frac{1}{2},1]$. Let $g = \begin{cases} 0 & [0,\frac{1}{2}] \\ 1 & [\frac{1}{2},1] \end{cases}$ and then

$\int_0^1 f g > 0 \Rightarrow$ contradiction. Similar story if $f<0$ on some subset of $[\frac{1}{2},1]$.

Projection on X: Let $f \in L^2$. We want to find

$$\inf_{w\in X} \| f - w\|^2 = \inf_{w\in X} \int_0^1 |f(x)-w(x)|^2 dx$$

Claim $w_0(x)$ (minimizer) is $w_0(x) = \begin{cases} 0 & [0,\frac{1}{2}] \\ f(x) & [\frac{1}{2},1] \end{cases}$

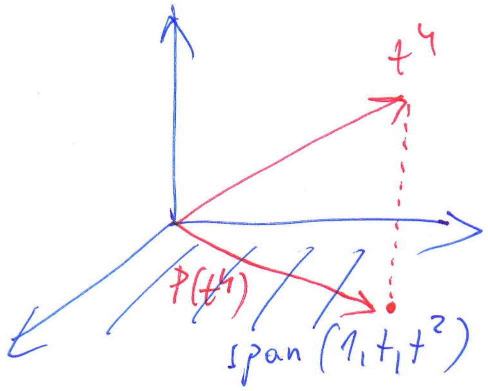
Let $w \in X$ arbitrary and compute:

$$\int_0^1 |f(x)-w(x)|^2 = \underbrace{\int_0^{1/2} |f(x)|^2}_{\text{does not depend on } w} + \underbrace{\int_{1/2}^1 |f(x)-w(x)|^2}_{\geq 0 \text{ and attains its minimum if } w=w_0}$$

In particular, we have checked that $\operatorname{dist}(f,X) = \left(\int_0^{1/2} |f|^2\right)^{1/2}$.

07 $\rightsquigarrow$ Big Homework.

08 We know that this is equivalent to computing projection of t^4 in $L^2(0,1)$ onto $\operatorname{span}(1,t,t^2)$.



By (04) we know that

$$\forall_{v\in \operatorname{span}(1,t,t^2)} \quad (v, t^4 - P(t^4)) = 0$$

Also $\exists_{a,b,c} \quad P(t^4) = a + bt + ct^2$.

$$\int_0^1 1\cdot(t^4-a-bt-ct^2)=0 \Rightarrow \frac{1}{5}-a-\frac{1}{2}b-\frac{1}{3}c=0$$

$$\int_0^1 t\,(t^4-a-bt-ct^2)=0 \Rightarrow \frac{1}{6}-\frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c=0$$

$$\int_0^1 t^2\,(t^4-a-bt-ct^2)=0 \Rightarrow \frac{1}{7}-\frac{1}{3}a-\frac{1}{4}b-\frac{1}{5}c=0$$

From this system of linear equations, we can find a, b, c .... ✓.

09 similar, you can practise...

Riesz Representation Theorem $\quad H=H^*$

$H=H^*$ means $\forall_{l\in H^*}\ \exists_{\tilde{l}\in H}\ \forall_{v\in H}\ l(v)=(\tilde{l},v)$.

We usually identify $\tilde{l}=l$ with equality which is slightly abused.

R1 $\quad f\mapsto \int_0^1 t^2 f(t)$ and $f\mapsto \int_0^1 e^t f(t)$ defines bounded linear functionals on $L^2(0,1)$. Hence, there are $g, h\in L^2(0,1)$ s.t. $\int_0^1 t^2 f(t) = \int_0^1 g(t)f(t)$ and $\int_0^1 e^t f(t) = \int_0^1 h(t) f(t)$ (namely: $h=e^t$, $g=t^2$)

It follows that $f \perp\!\!\!\perp e^t - h(t)$ and by 05 there are infinitely many such vectors.

(R2) By continuity and coercivity we have for some constants C, β

$$\beta \|u\|_H^2 \leq a(u,u) \leq C \|u\|_H^2 \quad (*)$$

hence, equivalence of norms and topologies. We have to check

(i) $\|u\|_a = \sqrt{a(u,u)}$ is a norm

(ii) $a(u,v)$ is a scalar product on H.

It will be helpful first to prove Cauchy-Schwartz inequality for $a(u,v)$, i.e $|a(u,v)|^2 \leq a(u,u)\,a(v,v)$ $\forall_{u,v \in H}$.

Proof: let $\lambda \in \mathbb{R}$. We compute

$$0 \leq a(u-\lambda v, u-\lambda v) = a(u,u) - 2\lambda a(u,v) + \lambda^2 a(v,v)$$

Let $\lambda = \frac{a(u,v)}{a(v,v)}$. Then $0 \leq a(u,u) - 2\frac{a(u,v)^2}{a(v,v)} + \frac{a^2(u,v)}{a(v,v)}$

$$\Rightarrow a(u,v)^2 \leq a(u,u)\,a(v,v).$$

From this we prove that $\|u\|_a$ satisfies triangle inequality.

$$\|u+v\|_a^2 = \|u\|_a^2 + \|v\|_a^2 + 2a(u,v) \leq$$

$$\leq \|u\|_a^2 + \|v\|_a^2 + 2\|u\|_a \|v\|_a = (\|u\|_a + \|v\|_a)^2$$

and the assertion follows.

Now, (i) and (ii) are trivial as a is bilinear symmetric and satisfies $(*)$.

$\square$.

R3 We apply R2. Now, originally ℓ is bounded linear functional on $(H, \langle\cdot,\cdot\rangle_H)$ but by equivalence of norms it is also a bounded linear functional on $(H, \langle\cdot,\cdot\rangle_A)$ (indeed, $\exists_C$

$|\ell(v)| \leq C\|v\|_H \Rightarrow |\ell(v)| \leq \tilde{C}\|v\|_A$).

By Riesz Representation Theorem, there is $\tilde{\ell}$ s.t. $\ell(v) = a(\tilde{\ell}, v)$ for all $v \in H$. Therefore, problem to be solved reads:

$$\forall_{v \in H} \quad a(u - \tilde{\ell}, v) = 0$$

Hence $u - \tilde{\ell} \in H^{\perp} = \{0\}$ and so $u = \tilde{\ell}$. By this, such u is unique.

$\{w \in H : \forall_{v \in H} (v, w) = 0\}$.

R4 Apply R3 with $a(u,v) = \int_0^1 u(t)v(t)e^t$ and $\underbrace{\ell(v) = \int_0^1 \sin(t)v(t)}_{\in (L^2)^*}$

By Hölder $|a(u,v)| \leq e\|u\|_{L^2}\|v\|_{L^2}$

Also $a(u,u) \geq \int_0^1 u^2(t)e^t \geq \int_0^1 u^2(t)\,dt = \|u\|^2_{L^2}$.

The assertion follows.

c_0 does not have complement in ℓ^∞

Let $(X, \|\cdot\|_X)$ be a normed space. We say that a linear subspace $Y \subset X$ has complement $Z \subset X$ if

(i) $X = Y \oplus Z$, i.e. $\forall_{x \in X} \ \exists!_{\substack{y \in Y \\ z \in Z}} \ x = y + z$

(ii) $Y \cap Z = \{0\}$

(iii) there is a bounded linear operator $P : X \longrightarrow X$ (called projection operator). (with range$(P) = Y$)

Note that (ii) guarantees uniqueness of the decomposition in (i).
We follow Stack Exchange post "Complement of c_0 in ℓ^∞".

C1 Let S be a countable infinite set. Then there is an uncountable "almost disjoint family" of infinite subsets of S, i.e. there is a family $\{A_i\}_{i \in I}$ of subsets of S s.t. $|I| \sim \mathbb{R}$, $|A_i| \sim \mathbb{N}$ and $\forall_{i \neq j}$ $A_i \cap A_j$ is finite. (Think of S as $\mathbb{Q}$ or $\mathbb{N}$).

Proof: Applying some isomorphism, one can think of S as $\mathbb{Q} \cap [0,1]$. Let I = irrational numbers on $[0,1]$. $\forall_{i \in I}$ we choose sequence of rational numbers from S $(= a_n^{(i)})$ s.t. $a_n^{(i)} \longrightarrow i$. Then we set $A_i = \{a_n^{(i)} : n \in \mathbb{N}\}$.

C2 Let $P : \ell^\infty \longrightarrow \ell^\infty$ be an operator such that $P(x) = 0$ for all $x \in c_0$. Then there is an infinite set of $\mathbb{N}$ such that $P(x) = 0$ for all x supported on A.

Proof: Applying isomorphism, we can think of A_i constructed in C1 as subsets of $\mathbb{N}$. Aiming at a contradiction, suppose that

$$\forall_{A_i \text{ (constr. in C1)}} \quad \exists_{\substack{x^i \in \ell^\infty \\ \text{supp on } A_i}} \quad P(x^i) \neq 0 \quad (\text{in part, } x^i \notin c_0).$$

normalize and assume $\|x^i\|_\infty = 1$

Since I is not countable, there is n s.t. $I_n = \{ i \in I : (Px^i)_n \neq 0 \}$ (note that $I = \bigcup_n I_n$). Similarly, $\exists_k$ s.t. $I_{n,k} = \{ i \in I : |(Px^i)_n| \geq \frac{1}{k} \}$ is not countable. We want to prove that $I_{n,k}$ is finite using that P is bounded.

To this end, let $J \subset I_{n,k}$ be finite and

$$y = \sum_{j \in J} \operatorname{sgn}\left[(Px^j)_n \right] \cdot x^j$$

Note that

$$(Py)_n = \sum_{j \in J} \underbrace{\operatorname{sgn}\left[(Px^j)_n \right] (Px^j)_n}_{\geq 0} \geq \frac{\#J}{k} \quad \text{as } J \subset I_{n,k}.$$

On the other hand, recall that x^i is supported on A_i and $A_i \cap A_j$ is finite whenever $i \neq j$. Hence, $y = f + z$

Note also $\|x^i\|_\infty \leq 1$

point with $\|z\|_\infty \leq 1$.

finitely supported from intersections $A_i \cap A_j$

Therefore $Py = Pf + Pz = Pz$ as f is finitely supported (i.e. $f \in c_0$) $\Rightarrow \|Py\|_\infty \leq \|P\| \|z\| \leq \|P\|$ as $\|z\| \leq 1$. $\Rightarrow \#J \leq k \|P\|$. Since RHS does not depend on $J \subset I_{n,k}$ it follows that $I_{n,k}$ is finite $\Rightarrow$ contradiction.

$\square$.

(C3) Conclusion:

Suppose there exists complement and let $Q: l^\infty \to l^\infty$ with range c_0 s.t. $\forall_{x \in c_0} Q(x) = x$. Then $P = I - Q$ satisfies C2. Therefore there is $A \subset \mathbb{N}$ infinite s.t. $Px = 0$ $\forall_{x \in l^\infty}$ supp on A

$\Rightarrow Qx = x \Rightarrow$ contradiction as sequence defined to be 1 on A and zero otherwise is not in c_0.

(B1) $\dim H = \infty$, H Hilbert space

(note: there is an error in formulation of Problem Set 5: "orthonormal countable basis)

($\Longrightarrow$) We want to prove it is separable. We claim that set $\sum_{i=1}^{n} q_i e_i$ for $n \in \mathbb{N}$, $q_i \in \mathbb{Q}$ and e_i = elements of basis is dense in H. Indeed, any $x \in H$ can be written as $x = \sum_{i=1}^{\infty} x_i e_i$ and the series is convergent in H. In particular, for any $\varepsilon > 0$, there is N s.t.

$$\left\| x - \sum_{i=1}^{N} x_i e_i \right\| \le \varepsilon/2$$

For each $i = 1, \dots, N$ choose q_i s.t. $|q_i - x_i| \le \varepsilon/2N$. Then

$$\left\| x - \sum_{i=1}^{N} q_i e_i \right\| \le \underbrace{\left\| x - \sum_{i=1}^{N} x_i e_i \right\|}_{\le \varepsilon/2} + \sum_{i=1}^{N} \underbrace{\| (x_i - q_i) e_i \|}_{\le \varepsilon/2N \text{ as } \|e_i\| = 1}$$

$\le \varepsilon$. (as desired).

($\Longleftarrow$) This is application of Zorn–Kuratowski Lemma. But we can also do that by Gram–Schmidt. Let $\{u_n\}_{n \in \mathbb{N}}$ be countable dense subset of H. By Gram–Schmidt we construct $\{e_m\}_{m \in \mathbb{N}}$ s.t.

$$\operatorname{span}\{e_1, e_2, \dots, e_m\} \supseteq \operatorname{span}\{u_1, \dots, u_n\} \quad (*)$$

(we have this inclusion as we can remove some linearly dependent vectors). We claim that $\{e_n\}_{n \in \mathbb{N}}$ is orthonormal SCHAUDER basis of H.

~~Let $x \notin \overline{\operatorname{span}\{u_1, \dots, u_n, \dots\}}$. Then, using projection on this closed subspace, $\langle x, u_i \rangle = 0 \Rightarrow \langle x, u_i \rangle = 0$. But since $\{u_i\}$ were dense we find some $u_i^x \to x$. Then $\langle x, x \rangle = \lim_{i \to \infty} \langle x, u_i^x \rangle \neq 0$ and $x \neq 0$~~

Indeed, consider series $S_x = \sum_{i=1}^{\infty} (x, e_i) e_i$ for some fixed $x \in H$ and we want to show $x = S_x$. $\boxed{v_i =: e_i}$ (notation...)

Note that $\langle x - S_x, e_j \rangle = 0 \ \forall j$ and so $x - S_x \perp \operatorname{span}(e_1, \ldots, e_j)$ and in particular $x - S_x \perp \overline{\operatorname{span}(e_1, \ldots, e_n, \ldots)}$.

On the other hand, we claim that $H = \overline{\operatorname{span}(e_1, \ldots, e_n, \ldots)}$.
For if not, write $G = \overline{\operatorname{span}(e_1, \ldots, e_n, \ldots)}$ and consider decomposition $H = G \oplus G^{\perp}$. Let $x \in G^{\perp}$. We have $(x, e_i) = 0$ and by $(*)$ also $(x, u_i) = 0$. Since $x = \lim u_i^x$ by density $(x, x) = \lim_{i \to \infty} (x, u_i^x) = 0$ and so $x = 0$ and so $H = G$

Therefore $x - S_x \perp H \Rightarrow x = S_x$.

Uniqueness of decomposition: Suppose $x = \sum_{i=1}^{\infty} x^i \cdot e_i$ and this series converges in H with $x^i \neq (x, e_i)$. But taking scalar product and using series convergence, $x^i = (x, e_i)$.

$\square$.

Conclusion: In separable Hilbert space, there is countable orthonormal set $\{e_i\}_{i=1}^{\infty}$ and $x = \sum_{i=1}^{\infty} (x, e_i) e_i$ as in linear Algebra.

Important Remark: Due to Per Enflo, there are Banach spaces which are separable but lack Schauder basis. Nevertheless, we know many Schauder basis (for c_0, ℓ^p).

(B2) If H is sep. Hilbert space, let $\{e_i\}$ be its basis as in B1.

For $x \in H$ we write $Tx = ((x, e_1), (x, e_2), \dots)$. We claim that T is

(0) ℓ^2-valued
(a) injective
(b) surjective
(c) bounded
(d) isometry

(a), (b) $\Rightarrow$ bijective; bijective + (c) $\Rightarrow$ bounded with bounded inverse

Ad (0): T is ℓ^2 valued by Bessel's inequality.

Ad (a): Suppose $Tx = 0$. Then $(x, e_i) = 0$ $\forall_i$. By B1 $x = 0$.

Ad (b): Let $y = (y_1, y_2, \dots, y_n, \dots) \in \ell^2$. Then $x = \sum y_i e_i \in H$ (this series converges as $y \in \ell^2$). Moreover $Tx = ((x, e_1), (x, e_2), \dots) = (y_1, \dots) = y$ as desired.

Ad (c): $\|Tx\|^2_{\ell^2} = \sum (x, e_i)^2 = \|x\|^2_H \Rightarrow \|Tx\|_{\ell^2} = \|x\|_H$

This also proves (d).

Conclusion: There are not so many separable Hilbert spaces.

(B3) We already know $\sum (x, e_i)^2 \leq \|x\|^2$. As $x = \sum (x, e_i) e_i$ we have $\|x\|^2 \leq \sum \|(x, e_i) e_i\|^2 = \sum (x, e_i)^2$ and conclusion follows.

(B4) Let us check that it is a Banach space. Let f_n be Cauchy sequence and let E_n be set of indices where $f_n(x) \neq 0$.

$$E_n = \{x \in \mathbb{R} : f_n(x) \neq 0\}, \quad E = \bigcup_n E_n = \text{countable}$$

Then $(f_n(x) : x \in E)$ with $f_n(x) = 0$ if $x \notin E_n$ is Cauchy in $L^2(\mathbb{N})$ $\Rightarrow$ $f_n(x) \longrightarrow f(x)$ in the considered space.

On the other hand, this space is not separable.

$$\| f - g \|^2 = \sum_{x \in \mathbb{R}} |f(x) - g(x)|^2$$

Let $f_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$. Then if $x_1 \neq x_2$ $\| f_{x_1} - f_{x_2} \| \geq \sqrt{2}$.

So we have constructed uncountable sequence separated with balls

$B(\cancel{x}f_{x_i}, \frac{\sqrt{2}}{4})$. $\square$.