Functional Analysis (WS 19/20), Problem Set 6

(Dual spaces, Hahn-Banach separation theorems and weak convergence)

Hahn-Banach Theorem (analytic form) Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ be a linear subspace. Let $p : X \to \mathbb{R}$ be such that

$$p(x+y) \le p(x) + p(y), \qquad p(tx) = tp(x)$$

for all $x, y \in X$ and $t \ge 0$. Finally, suppose that $g : M \to \mathbb{R}$ is a linear functional and $g(x) \le p(x)$ for all $x \in M$. Then, there exists a linear functional $f : X \to \mathbb{R}$ such that $f(x) = g(x)$ on M and $f(x) \le p(x)$ for all $x \in X$.

See also Problem H1 for a simpler version of this result.

Hahn-Banach Theorem (geometric form) Let $(X, \|\cdot\|)$ be a normed space. Let $A, B \subset X$ be nonempty, convex and disjoint sets.

1. If A is open, there exists $\varphi \in X^*$ and λ such that

$$\varphi(x) < \lambda \le \varphi(y)$$

for all $x \in A$ and $y \in B$. We say that hyperplane $\{x \in X : \varphi(x) = \lambda\}$ separates A and B.

2. If A is closed and B is compact, there exists $\varphi \in X^*$ and λ_1, λ_2 such that

$$\varphi(x) < \lambda_1 < \lambda_2 < \varphi(y)$$

for all $x \in A$ and $y \in B$. Let $\lambda = \frac{\lambda_1 + \lambda_2}{2}$. We say that hyperplane $\{x \in X : \varphi(x) = \lambda\}$ separates strictly A and B.

Dual spaces characterization

D1. ♣ Let H be a Hilbert space. Recall from the lecture that $H = H^*$ in the sense of isometric isomorphism. Write explicitly this isomorphism.

D2. ♣ Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Recall from the lecture that for $1 \le p < \infty$, $(L^p)^* = L^q$ in the sense of isometric isomorphism (here $1/p + 1/q = 1$). Write explicitly this isomorphism.

D3. Prove that the map $T : l^1 \to (c_0)^*$ given with

$$(Ty)(x) = \sum_{i=1}^{\infty} x_i y_i$$

is well-defined, injective, surjective and isometry (i.e. $\|y\|_{l_1} = \|Ty\|_{(c_0)^*}$). Conclude that $(c_0)^* = l_1$.

Hahn-Banach Theorem and its applications

H1. ♣ Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ be a linear subspace. Let $g \in M^*$. Prove that there is a bounded linear functional $f \in X^*$ such that $g(x) = f(x)$ for $x \in M$ and $\|f\|_{X^*} = \|g\|_{M^*}$.

H2. Let $I : c_0 \to c_0$ be the identity operator on c_0. Prove that P cannot be extended to l^∞.[1]

H3. Let $(X, \|\cdot\|)$ be a normed space and $x_0 \in X$. Prove that there is $\varphi \in X^*$ such that

$$\varphi(x_0) = \|x_0\|^2 \text{ and } \|\varphi\| = \|x_0\|.$$

H4. ♣ Let $(X, \|\cdot\|)$ be a normed space. Prove that

$$\|x\| = \sup_{f \in X^* : \|f\| \leq 1} f(x)$$

and the supremum above is attained. Moreover, if X^* is separable, prove that the supremum above can be taken over countable family of linear functionals $f \in X^*$ such that $\|f\| \leq 1$.

H5. ♣ Let $(X, \|\cdot\|)$ be a normed space. Prove that if $\varphi(x_1) = \varphi(x_2)$ for all $\varphi \in X^*$ then $x_1 = x_2$.

H6. ~~Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \|\cdot\|)$ be a random variable. Suppose that E^* is separable. Prove that $\|X\|$ is a random variable again (i.e. it is measurable).~~

H7. Let $(E, \|\cdot\|)$ be a Banach space and $A \subset E$ be its subset. Suppose that for every $f \in E^*$, the set

$$f(A) = \{f(x) : x \in A\}$$

is bounded in $\mathbb{R}$. Prove that A is a bounded set in E (i.e. one can find a ball $B(0, R)$ for some $R > 0$ such that $A \subset B(0, R)$).

H8. Consider $L^p(\Omega, \mathcal{F}, \mu)$ with $1 \leq p < \infty$ and $1/p + 1/q = 1$. Prove that

$$\|f\|_p = \sup_{g \in L^q : \|g\|_q \leq 1} \int_X f(x)g(x)d\mu(x),$$

H9. Prove that $l^1 \subset (l^\infty)^*$ but $(l^\infty)^* \neq l^1$. *Hint:* Consider $c \subset l^\infty$.

H10. ♣ Let E be a normed space and $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Prove that there is $\varphi \in E^*$ such that $\varphi \neq 0$, $\|\varphi\| = 1$ and $\varphi(x) = 0$ for all $x \in F$.

H11. Let E be a normed space and $F \subset E$ be a linear subspace such that for all $\varphi \in E^*$

$$\forall_{x \in F}\ \varphi(x) = 0 \implies \varphi = 0.$$

Prove that F is dense in E.

H12. Let X be a vector space (not necessarily normed or Banach) over $\mathbb{R}$. Let φ, φ_1, ..., φ_k be linear functionals on $\mathbb{R}$ (i.e. linear maps from X to $\mathbb{R}$). Suppose that

$$(\forall_{i=1,\dots,k}\ \varphi_i(v) = 0) \implies \varphi(v) = 0.$$

Prove that φ is a linear combination of $\varphi_1, ..., \varphi_k$, i.e. there are real numbers $\lambda_1, ..., \lambda_k$ such that $\varphi = \sum_{n=1}^k \lambda_n \varphi_n$. *Hint:* Study $F(x) = (\varphi_1(x), ..., \varphi_k(x), \varphi(x))$.

H13. ♣ (**Riesz Lemma**) Let $(X, \|\cdot\|)$ be a normed space and $M \subset X$ a closed (strictly contained) subspace. Prove that for any $\alpha \in (0, 1)$ there is $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x, M) \geq \alpha$.

[1] Kakutani Theorem (1940) asserts that every operator on the closed subspace M in a Banach space $(X, \|\cdot\|)$ can be extended if and only if X is a unitary space (its norm satisfies paralellogram identity).

H14. Prove that if X is finite dimensional, one can obtain Riesz Lemma for $\alpha = 1$. Prove that this is not possible, in general, for infinite dimensional X (study $X = l^\infty$).

H15. ♣ (**compactness of the ball**) Use Riesz Lemma to prove that if $(X, \|\cdot\|)$ is infinite dimensional space, ball $B_X = \{x \in X : \|x\| \leq 1\}$ is not compact.

H16. In the following Problem we will see that in infinite dimensional setting, *something* has to be assumed about two convex sets so that they can be separated (in finite dimensional case, convexity of both sets is sufficient). Let $E = l^1$ with its usual norm and consider two subsets:

$$X = \left\{x \in l^1 : x_{2n} = 0 \text{ for all } n \geq 1\right\}$$

$$Y = \left\{y \in l^1 : y_{2n} = \frac{1}{2^n} y_{2n-1} \text{ for all } n \geq 1\right\}.$$

(a) Check that X and Y are closed linear spaces in l^1. Verify that $\overline{X + Y} = E$.

(b) Consider sequence c defined with $c_{2n-1} = 0$ and $c_{2n} = \frac{1}{2^n}$. Check that $c \notin X + Y$.

(c) Set $Z = X - c$ and check that $Y \cap Z = \emptyset$. Can one separate Y and Z?

Introduction to weak convergence

Let $(E, \|\cdot\|)$ be a Banach space. We say that sequence $(x_n)_{n\geq 1} \subset E$ converges weakly to $x \in E$ if for every $\varphi \in E^*$ we have $\varphi(x_n) \to \varphi(x)$. We write $x_n \rightharpoonup x$.

W1. ♣ Write explicitly, using representation theorems, what does it mean to converge weakly in L^p (for $1 \leq p < \infty$) and H where H is a Hilbert space.

W2. ♣ Prove that weak limits are unique: if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ then $x = y$.

W3. ♣ Prove that sequences converging weakly are bounded, i.e. if $x_n \rightharpoonup x$ then there is a constant C such that $\|x_n\| \leq C$ where C does not depend on $n \in \mathbb{N}$. Moreover, prove the bound

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

W4. ♣ Prove that if $x_n \to x$ then $x_n \rightharpoonup x$.

W5. Prove that $\sin(nx) \rightharpoonup 0$ but $\sin^2(nx) \rightharpoonup \frac{1}{2}$ in $L^p(0, 2\pi)$ for $1 < p < \infty$. Hence, nonlinearities do not preserve weak limits. *Remark:* Unfortunately, one can show much more: if F is a function such that $F(x_n) \rightharpoonup F(x)$ for all $x_n \rightharpoonup x$, then F is an affine function.

W6. Prove that if $x_n \rightharpoonup x$ and $f_n \to f$ in E^* then $f_n(x_n) \to f(x)$ as $n \to \infty$.

W7. (~~**Riesz**~~ **Mazur Lemma**) Let $C \subset E$ be a convex set. Prove that C is closed for convergence in norm if and only if C is closed for weak convergence. *Hint*: Hahn-Banach.
<u>C is closed for convergence in norm</u> if for any $\{x_n\}_{n\geq 1}$ such that $x_n \to x$ it follows that $x \in C$. This is exactly the same as statement that C is closed in E.
<u>C is closed for weak convergence</u> if for any $\{x_n\}_{n\geq 1}$ such that $x_n \rightharpoonup x$ it follows that $x \in C$.

W8. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that f attains its minimum in some point $x \in [0, 1]$. Moreover, prove that lowersemicontinuity of f is sufficient.

W9. ♣ (**Banach-Alaoglu-Bourbaki Theorem, special case**) Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H. Prove that $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging weakly to some $x \in H$. [2]

W10. Let $(H, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Prove that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = 1$ and $x_n \rightharpoonup 0$.[3]

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[2] This result is probably the most important one in Functional Analysis and holds for more general spaces.

[3] In fact, if E is a uniformly convex Banach space (note that Hilbert spaces are always uniformly convex) one can easily prove (using that a closed ball is also weakly closed) that $x_n \to x$ if and only if $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$.