Functional Analysis Problem Set 6

Dual spaces, Hahn–Banach Thms and weak conv.

(D1) $\underset{f\in H^*}{\forall}\ \underset{\tilde f\in H}{\exists!}\ \underset{u\in H}{\forall}\ f(u) = \langle \tilde f, u\rangle$

We can write $H \ni \tilde f \longmapsto f = \langle \tilde f, \cdot \rangle \in H^*$.

(D2) $\left(L^p(\Omega, \mathcal{F}, \mu)\right)^* = L^q(\Omega, \mathcal{F}, \mu)$ $\quad \frac{1}{p}+\frac{1}{q}=1.\ ,\ 1\leq p<\infty.$

Special cases: $(L^2)^* = L^2$, $(L^1)^* = L^\infty$.

We can write $\underset{f\in (L^p)^*}{\forall}\ \underset{\tilde f\in L^q}{\exists!}\ \underset{g\in L^p}{\forall}\ f(g) = \int_X g(x)\tilde f(x)\,d\mu(x)$.

or $L^q \ni \tilde f \longmapsto \int_X \boxed{\bullet}\,\tilde f(x)\,d\mu(x)$.

(D3) $\rightsquigarrow$ Big Homework 3

more practical version of extension theorem.

(H1) Concider $p(x) = \|x\|\,\|g\|_{M^*}$ in analytic form of Hahn–Banach Thm.

Clearly $p(x+y)\leq p(x)+p(y)$, $p(tx) = tp(x)\ \forall_{t>0}$. $\Rightarrow \exists_{f: X\to\mathbb{R}}$

$\forall_{x\in X}\ f(x) \leq \|x\|\cdot\|g\|_{M^*}$. We want to bound $\sup_{\|x\|\neq 0} \frac{|f(x)|}{\|x\|}$.

$$\frac{|f(x)|}{\|x\|} \leq \begin{cases} \text{if } f(x) \text{ is positive}: \leq \|g\|_{M^*} \\ \text{if } f(x) \text{ is negative}: -\frac{f(x)}{\|x\|} = \frac{f(-x)}{\|x\|} = \frac{f(-x)}{\|-x\|} \leq \|g\|_{M^*} \end{cases}$$

As $f = g$ on M, $\|f\|_{X^*} = \|g\|_{M^*}$.

H2 Consider identity operator on c_0. $P: c_0 \to c_0$ and $P = Id$.

Suppose P can be extended to ℓ^∞ (note that c_0 is closed subspace of ℓ^∞). We claim that $\ell^\infty = c_0 \oplus \text{range}(I-P)$. Clearly, P is projection that is continuous. We should just check that

$$c_0 \cap \text{range}(I-P) = \{0\}$$

Jędrek's solution later...

To this end, suppose $x \in c_0$, $x \in \text{range}(I-P)$. Then, $\exists_y\ x = y - Py$

But $x \in c_0$, $Py \in c_0 \Rightarrow y = x + Py \in c_0 \Rightarrow Py = y \Rightarrow x = y - Py = 0$.

Therefore, c_0 would have complement in $\ell^\infty \Rightarrow$ contradiction.

Note that in Hilbert spaces any operator det. initially on closed subspace can be extended. $\square$.

H3 We apply **H1** with $M = \text{lin}\{x_0\}$. We set $\varphi(tx_0) = t\|x_0\|^2$.

Then $\varphi(x_0) = \|x_0\|^2$ and $\|\varphi\| = \sup_{\mathbb{R}^* \ni tx_0 \neq 0} \frac{\varphi(tx_0)}{\|tx_0\|} = \sup_t \frac{t\|x_0\|^2}{t\|x_0\|} =$

$= \|x_0\|$. Extend with **H1**. We write φ_{x_0} for such functional constructed for a given x_0.

H4 Clearly $f(x) \leq \|f\|\|x\|$ and so, $\sup_{f \in X^*, \|f\| \leq 1} f(x) \leq \|x\|$.

By **H3** this sup is attained for $\frac{\varphi_{x_0}}{\|x_0\|}$ (note that $\left\|\frac{\varphi_{x_0}}{\|x_0\|}\right\| = 1$)

Let X^* be separable and $\{f_k\}_{k=1}^\infty$ be countable dense subset of X^* (or better: $\underbrace{X^* \cap B_1(0)}_{\text{unit ball}}$). Clearly $\sup_{f \in X^*, \|f\| \leq 1} f(x) \geq \sup_k f_k(x)$.

To prove opposite inequality, fix $x \in X$, $\varepsilon > 0$. For each f, there is f_k s.t. $\|f_k - f\|_{X^*} \leq \varepsilon / \|x\|$ Then $f(x) \leq f(x) - f_k(x) + f_k(x) \leq$

$\leq \|f_k - f\|_{X^*}\|x\| + f_k(x) \leq \varepsilon + \sup_k f_k(x)$

Take sup on the (LHS) to obtain $\sup_{f \in X^*, \|f\| \leq 1} f(x) \leq \varepsilon + \sup_k f_k(x)$.

As $\varepsilon > 0$ is arbitrary $\Rightarrow \sup_{f \in X^*, \|f\| \leq 1} f(x) \leq \sup_k f_k(x)$ and equality is established.

H5 $\varphi(x_1) = \varphi(x_2)$ $\forall_{\varphi \in X^*}$ $\Rightarrow$ $\varphi(x_1 - x_2) = 0$ $\forall_{\varphi \in X^*}$ <u>but</u>

$$\|x_1 - x_2\| = \sup_{\varphi \in X^* : \|\varphi\| \leq 1} \varphi(x_1 - x_2) = 0 \Rightarrow x_1 = x_2.$$

H6 $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (E, \|\cdot\|)$ $\quad E^*$ separable $\Rightarrow$ there is $\{\varphi_k\} \subset E^*$ with $\|\varphi_k\| \leq 1$ such that

$$\|X\| = \sup_k \varphi_k(X)$$

$\|$

countable supremum of measurable maps (even continuous) $\Rightarrow$ measurable.

H7 $\rightsquigarrow$ Big Homework 3

H8 We know that if $x \in E$, $\|x\| = \sup_{f \in E^*, \|f\|_{E^*} \leq 1} f(x)$

For $E = L^p$, $E^* = L^q$ $\left(1 \leq p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1\right)$ and map between $(L^p)^*$ and L^q is isometric (i.e. it preserves the norm).

Hence, if $f \in L^p$

$$\|f\|_{L^p} = \sup_{g \in L^q, \|g\|_{L^q} \leq 1} \int_X fg \, d\mu . \qquad \square$$

(H9) $\ell^1 \subset (\ell^\infty)^*$: indeed if $x = (x_1, \dots, x_n, \dots) \in \ell^1$ we define functional $\ell^\infty \ni y \longmapsto \sum_{k \geq 1} y_k x_k$. By Hölder

$$\Big| \sum_{k \geq 1} x_k y_k \Big| \leq \|x\|_1 \, \|y\|_\infty \qquad \text{(or by simple estimates).}$$

so this defines) a bounded linear functional on ℓ^∞. Moreover, the map

$$\ell^1 \ni x \longmapsto \Big(y \longmapsto \sum_{k \geq 1} y_k x_k \Big) \in (\ell^\infty)^*$$

is injective: as the map is linear it is sufficient to check that $x = 0$ corresponds to zero functional and this is indeed true.

To prove $\ell^1 \neq (\ell^\infty)^*$, it is sufficient to find $f \in (\ell^\infty)^*$ such that $f(y)$ cannot be represented as $\sum_{k \geq 1} y_k x_k$ for some $x \in \ell^1$.

Recall: $c \subset \ell^\infty$ = closed subspace of bounded sequences in ℓ^∞.

Define $\varphi(y) = \lim_{k \to \infty} y_k$ for $y \in c$. Clearly $|\varphi(y)| \leq \|y\|_\infty$.

Extend φ by Hahn-Banach to ℓ^∞. Call this extension $\tilde{\varphi} \in (\ell^\infty)^*$. If $\varphi(y) = \sum_{k \geq 1} x_k y_k$ for some $x \in \ell^1$ then

$$\underset{\substack{\uparrow \\ \text{limit of } e^k \\ \text{at infinity}}}{0} = \varphi(\underset{\substack{\uparrow \\ \text{basis vector}}}{e^k}) = x_k \Rightarrow x = 0.$$

Contradiction as $\varphi \neq 0$

for instance $\varphi((1,1,1,\dots)) = 1$).

$\square$.

Geometric versions of Hahn-Banach

$(E, \|\cdot\|)$ normed space

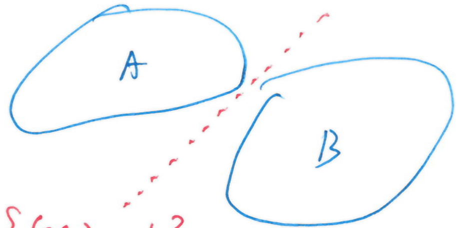
- $A, B \subset E$, one of them is <u>open</u>, convex, nonempty, disjoint.

$\{\varphi(x) = \lambda\}$

$$\sup_{a \in A} \varphi(a) \leqslant \lambda \leqslant \inf_{b \in B} \varphi(b)$$

$\boxed{\text{SEPARATION}}$

- $A, B \subset E$, <u>A closed</u>, B <u>compact</u>, both convex, nonempty, disjoint



$\{\varphi(x) = \lambda\}$

$$\sup_{a \in A} \varphi(a) < \lambda < \inf_{b \in A} \varphi(b)$$

$\boxed{\text{STRICT SEPARATION}}$

More practical version of H-B geometric theorems is given below:

(H10) Let $x_0 \in E \setminus \overline{F}$ and use H-B strict separation with closed set $\overline{F}$ and compact $\{x_0\}$. Then $\exists_{\varphi \in E^*}, \exists_{\lambda \in \mathbb{R}}$ s.t.

$$\sup_{f \in F} \varphi(f) < \lambda < \varphi(x_0)$$

In particular, $\forall_{f \in F}$ $\varphi(f) < \lambda < \varphi(x_0)$. As F is subspace $\varphi(f) < \lambda \Rightarrow \varphi|_F = 0$ (we can scale elements in F).

But then $0 < \lambda < \varphi(x_0)$. Finally, we take $\dfrac{\varphi(x)}{\|\varphi\|}$.

$$\Downarrow$$
$$\varphi \neq 0$$

$\square$.

H13 Riesz Lemma $\rightsquigarrow$ small homework

H15 $(X, \|\cdot\|)$ has $\dim X = \infty \Rightarrow B_X$ is not compact.

We will find sequence without any converging subsequence. Choose e_1 with $\|e_1\|=1$. Define $M_1 = \operatorname{span}\{e_1\}$ – closed subspace. Then, using Riesz Lemma, find e_2 s.t. $\operatorname{dist}(e_2, M_1) \geq \frac{1}{2}$, $\|e_2\|=1$. Proceeding by induction we obtain $\{e_k\}_{k\in\mathbb{N}}$ s.t. $\|e_k\|=1$ and $\|e_j - e_k\| \geq \frac{1}{2}$ whenever $j \neq k$. Clearly, this sequence cannot have convergent subsequence (for instance: it is not Cauchy).

H11 Suppose that F is not dense in E, i.e. $\overline{F} \neq E$ (but still $\overline{F} \subset E$). This means that there is φ s.t. $\varphi|_F = 0$ but $\varphi \neq 0$. Contradiction.

Remark: This fact is used to prove that some subspaces are dense in normed spaces.

H12 $\rightsquigarrow$ exercise in Big Homework 3 for Hahn-Banach Thm

H14 Finite dimensional case:

$$\underset{1>\alpha>0}{\forall} \ \underset{x_\alpha \ \|x_\alpha\|=1}{\exists} \quad \text{s.t. } \operatorname{dist}(x_\alpha, M) \geq \alpha \iff \underset{y\in M}{\forall} \ \|x_\alpha - y\| \geq \alpha$$

Take sequence $\alpha_n = 1 - \frac{1}{n}$ and let x_n be the sequence obtained from Riesz Lemma (for α_n). Since X is finite dimensional, ball B_X is compact and there is subsequence of $\{x_n\}$ denoted with $\{x_{n_k}\}$ converging in norm, i.e. $\underset{x}{\exists}\ x_{n_k} \longrightarrow x$, $\|x\|=1$.

For fixed $y \in M$ $\|x_{n_k} - y\| \geq \alpha_{n_k}$. Passing with $n_k \longrightarrow \infty$ we deduce $\|x-y\| \geq 1 \ \underset{y\in M}{\forall} \Rightarrow \operatorname{dist}(x, M) \geq 1$ as desired.

Riesz Lemma with $\alpha = 1$ cannot be obtained in $\dim X = \infty$ case (in general).

Let $X = \ell^\infty$.

In this case construction is pretty tricky. We proceed with a different example that is given in Special Problems (see Special Problem 4).

H15

(A) Indeed, X and Y are closed as convergence in ℓ^1 implies convergence of sequence elements.

To check that $\overline{X+Y} = \ell^1$, we prove that $e_i = (0,0,\dots,0,\underset{i\text{-th}}{1},0,\dots)$ belongs to $X+Y$ (this is sufficient as we know that sequence $\{e_i\}_i$ forms Schauder basis of ℓ^1 so $\overline{\mathrm{span}(e_1, e_2, \dots)} = \ell^1$.

Note that $e_1, e_3, e_5, \dots \in X$ so $e_1, e_3, e_5 \in X+Y$ (we take element from Y to be 0. Note that the sequence $y^n\{y_i^n\}$

$$y_i^n = \begin{cases} 1 & \text{on } (2n-1) \text{ position} \\ \frac{1}{2^n} & \text{on } 2n \text{ position} \\ 0 & \text{otherwise} \end{cases} \in Y.$$

Then
$$e_{2n} = \underbrace{2^n \cdot y^n}_{\in Y} - \underbrace{2^n \cdot \underbrace{e_{2n-1}}_{\in X}}_{\in X} \in X+Y.$$

✓.

(B) Sequence c is defined with $c_{2n-1} = 0$, $c_{2n} = \frac{1}{2^n}$.

Suppose $c = x+y$ where $x \in X$, $y \in Y$.

$c_{2n} = x_{2n} + y_{2n} \Rightarrow c_{2n} = y_{2n} = \frac{1}{2^n} \Rightarrow y_{2n-1} = 1 \Rightarrow y \notin \ell^1$.
contradiction

(C) $Z = X - c$. $Z \cap Y = \emptyset$: Let $u \in X-c$, $u \in Y$. $\Rightarrow$

$\exists_x\ u = x - c \Rightarrow c = \underbrace{x}_{\in X} - \underbrace{u}_{\in Y} \Rightarrow c \in X+Y$ contradiction

(D) Suppose there is $\varphi \in (\ell^1)^*$ s.t. for some $\lambda \in \mathbb{R}$

$$\varphi(z) \leq \lambda \leq \varphi(y) \quad \forall_{z \in Z} \ \forall_{y \in Y}$$

As $Z = X - c$ we can write

$$\varphi(x) - \varphi(c) \leq \lambda \leq \varphi(y) \quad \forall_{x \in X} \ \forall_{y \in Y}$$

But this shows $\varphi(x) \leq \lambda + \varphi(c) \quad \forall_{x \in X}$ $\Rightarrow$ $\varphi|_X = 0$

$\varphi(y) \geq \lambda \quad \forall_{y \in Y}$ $\quad \varphi|_Y = 0$

$\Rightarrow \varphi|_{X+Y} = 0$. As $\overline{X+Y} = \ell^1$ we have $\varphi|_{\ell^1} = 0$

$\Rightarrow$ contradiction as φ is 0 functional.

$\square$.

Introduction to weak convergence

In H15 we have seen that ball in infinite dimensional space $(E, \|\cdot\|)$ is not compact. Compactness is a crucial property in analysis allowing to extract patterns and regularity from bounded sets. Lack of compactness was a significant problem in XX century in PDEs, Optimal Control where many proofs usually starts with extraction of converging subsequence (just think about proving that the continuous $f: [0,1] \to \mathbb{R}$ attains its minimum).

The main idea to deal with that problem was to WEAKEN norm topology which seemed to be to strong. This means that if topology has less open sets, it is easier for the sequence to converge. As we will see, this procedure allows to recover some weak concept of compactness, a result due to many mathematicians (Alaoglu, Banach, "Bourbaki"). $\square$

The relevant definition goes as follows:

Definition (weak convergence)

Let $(X, \|\cdot\|)$ be a Banach space. We say that x_n converges weakly to x (we write: $x_n \rightharpoonup x$) if for any $\varphi \in X^*$ we have:

$$\varphi(x_n) \longrightarrow \varphi(x) \quad \text{as } n \to \infty$$

(or $\varphi(x_n - x) \to 0$). $\square$.

W1 In Hilbert spaces $(H, \langle \cdot ; \cdot \rangle)$: $x_n \rightharpoonup x$ if and only if

$$\forall_{h \in H} \quad \langle h, x_n \rangle \longrightarrow \langle h, x \rangle.$$

In L^p spaces with $1 \leq p < \infty$: $f_n \rightharpoonup f$ if and only if

$$\forall_{g \in L^q} \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \quad \int_X f_n(x) g(x) \longrightarrow \int_X f(x) g(x).$$

W2 If

$$x_n \rightharpoonup x \Rightarrow \varphi(x_n) \longrightarrow \varphi(x) \quad \forall_{\varphi \in E^*}$$
$$x_n \rightharpoonup y \Rightarrow \varphi(x_n) \longrightarrow \varphi(y) \quad \forall_{\varphi \in E^*}$$

$$\Rightarrow \varphi(x) = \varphi(y) \quad \forall_{\varphi \in E^*} \Rightarrow x = y. \qquad \square$$

W3 Suppose $x_n \rightharpoonup x$, i.e. for all $\varphi \in E^*$ $\varphi(x_n) \longrightarrow \varphi(x)$.

Clearly, we have to apply Banach-Steinhaus Theorem. To this end, let $T_n(\varphi) = \varphi(x_n)$, $T_n : E^* \longrightarrow \mathbb{R}$ (by the way, $T_n \in E^{**}$)

We have $|T_n(\varphi)| \leq |\varphi(x_n)|$ ← for fixed $\varphi \in E^*$, this sequence is bounded as it is convergent.

Hence if $\varphi \in E^*$ is fixed, $\sup_n |T_n(\varphi)| \leq C_\varphi$. In particular, by BST, $\sup_n \|T_n\| \leq C$ i.e. $\sup_n \sup_{\|\varphi\| \leq 1} |\underbrace{T_n(\varphi)}_{= \varphi(x_n)}| \leq C$

independent constant

We know that $\|x_n\| = \sup_{\|\varphi\| \leq 1} \varphi(x_n)$. Therefore, $\sup_n \|x_n\| =$

$= \sup_n \|x_n\| = \sup_n \sup_{\|\varphi\| \leq 1} \varphi(x_n) \leq C$ as desired. Similarly,

This trick with sup and liminf is useful in Analysis

$$\|x\| = \sup_{\|\varphi\| \leq 1} \varphi(x) = \sup_{\|\varphi\| \leq 1} \liminf_{n \to \infty} \varphi(x_n) \leq \liminf_{n \to \infty} \sup_{\|\varphi\| \leq 1} \varphi(x_n) \leq$$

$$\leq \liminf_{n \to \infty} \sup_{\|\varphi\| \leq 1} \|\varphi\| \|x_n\| \leq \liminf_{n \to \infty} \|x_n\| \qquad \checkmark.$$

W4 Let $x_n \rightharpoonup x$. Then $\varphi(x_n) - \varphi(x) \to 0$ as

$|\varphi(x_n - x)| \leq \|\varphi\| \, \|x_n - x\| \to 0$. Therefore $x_n \rightharpoonup x$. $\square$.

W5 We need to prove $\int_0^{2\pi} f(x)\sin(nx)\,dx \to 0$ as $n \to \infty$ for all $f \in L^{p'}(0, 2\pi)$ $\left(\frac{1}{p'} + \frac{1}{p} = 1\right)$.

First, suppose f is a simple function, $f = \sum_{i=1}^{m} a_i \mathbb{1}_{[c_i, c_{i+1}]}$, $c_1 = 0$, $c_{m+1} = 2\pi$.

Then, $\int_0^{2\pi} f(x)\sin(nx)\,dx = \sum_{i=1}^{m} a_i \int_{c_i}^{c_{i+1}} \sin(nx) = \sum_{i=1}^{m} \frac{a_i}{n}\left[\cos(nc_{i+1}) - \cos(nc_i)\right]$

$$= \frac{1}{n} \underbrace{\sum_{i=1}^{m} a_i \left[\cos(nc_{i+1}) - \cos(nc_i)\right]}_{\text{bounded constant}} \to 0 \text{ as } n \to \infty.$$

Fix $\varepsilon > 0$.

For general $f \in L^{p'}(0, 2\pi)$, we find $f_\varepsilon \in L^{p'}(0, 2\pi)$, f_ε – simple function like above, such that $\|f_\varepsilon - f\|_{p'} \leq \varepsilon$.

Then,

$$\limsup_{n\to\infty} \left|\int_0^{2\pi} f(x)\sin(nx)\,dx\right| \leq \underbrace{\limsup_{n\to\infty}\left|\int_0^{2\pi} (f(x) - f_\varepsilon(x))\sin(nx)\,dx\right|}_{\text{Hölder: } \|f - f_\varepsilon\|_{p'} \|\sin(nx)\|_p \leq C\varepsilon}$$

$$+ \underbrace{\limsup_{n\to\infty}\left|\int_0^{2\pi} f_\varepsilon(x)\sin(nx)\,dx\right|}_{= 0 \text{ by the previous fact for simple function.}}$$

Hence, $\limsup_{n\to\infty} \left|\int_0^{2\pi} f(x)\sin(nx)\,dx\right| \leq C\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the conclusion follows.

To study weak limit of $\sin^2(nx)$, we remind that

$$\int \sin^2(x) = \frac{x}{2} - \frac{1}{4}\sin(2x) + \text{const}$$

We repeat computation for simple functions: $f = \sum_{i=1}^{m} a_i \mathbb{1}_{[c_i, c_{i+1}]}$, $c_1 = 0$, $c_{m+1} = 2\pi$.

$$\int_0^{2\pi} \sin^2(nx) f(x)\,dx = \sum_{i=1}^{m} a_i \int_{c_i}^{c_{i+1}} \sin^2(nx)\,dx =$$

$$= \sum_{i=1}^{m} \frac{a_i}{n}\left[\frac{nc_{i+1} - nc_i}{2} - \frac{\sin(2nc_{i+1}) - \sin(2nc_i)}{4}\right]$$

$$= \underbrace{\frac{1}{2}\sum_{i=1}^{m} a_i (c_{i+1} - c_i)}_{=\int f} + \frac{1}{n}\underbrace{\sum_{i=1}^{m} a_i \left(\sin(2nc_i) - \sin(2nc_{i+1})\right)}_{\text{bounded number}}$$

$$\longrightarrow \int_0^{2\pi} \frac{1}{2} f(x)\,dx \quad \Rightarrow \quad \int_0^{2\pi}\left[\sin^2(nx) - \frac{1}{2}\right] f(x) \longrightarrow 0$$

Then, we approximate as in the case of $\sin(x)$ proving that $\sin^2(nx) \rightharpoonup \frac{1}{2}$

Remark: The VERY important observation here is that $\int_0^{2\pi} \sin x = 0$, $\int_0^{2\pi} \sin^2 x = \frac{1}{2}$.

It can be checked [Special Problems] that for periodic functions $f(x)$ rescalings $f(nx)$ converges to averages.

Remark: Note that $\sin(nx) \rightharpoonup 0$, $\sin^2(nx) \rightharpoonup \frac{1}{2}$ and $(0)^2 \neq \frac{1}{2}$

so it is not true in general that $x_n \rightharpoonup x \Rightarrow F(x_n) \rightharpoonup F(x)$.

Actually, it can be proved that this is only true if F is an affine function [Special Problems].

(W5) $x_n \rightharpoonup x$ in E $\Rightarrow$ $\{x_n\}$ is bounded in E by W3

$f_n \to f$ in E^*

$$|f_n(x_n) - f(x)| \leq \underbrace{|f_n(x_n) - f(x_n)|}_{\substack{\leq \left(\sup_n \|x_n\|\right) \|f_n - f\| \\ \leq C \|f_n - f\| \to 0}} + \underbrace{|f(x_n) - f(x)|}_{\to 0 \text{ as } x_n \rightharpoonup x.}$$

(W6) (small homework)

For convex set $C \subset E$ we have equivalence:

- C is strongly closed: $\{x_n\} \subset C$ and $x_n \to x \Rightarrow x \in C$
- C is weakly closed: $\{x_n\} \subset C$ and $x_n \rightharpoonup x \Rightarrow x \in C$

Hint: Geometric Hahn-Banach (note carefully that geometric HB applies to strongly closed sets, not the weakly ones!

Remark: This result can be used to prove that if $F : E \to \mathbb{R}$ is convex and strongly lower semi-continuous function, i.e.

$$x_n \to x \Rightarrow F(x) \leq \liminf_{n \to \infty} F(x_n)$$

then F is weakly lowersemicontinuous function, i.e.

$$(*) \; x_n \rightharpoonup x \Rightarrow F(x) \leq \liminf_{n \to \infty} F(x_n).$$

In particular, $(*)$ holds if F is strongly continuous and convex. Property $(*)$ is usually sufficient in many applications like minimization of functionals, see SPECIAL PROBLEMS.

W7 If $f:[0,1]\to\mathbb{R}$, let $\alpha = \inf_{x\in[0,1]} f(x)$ (possibly $-\infty$). Let x_n be a sequence s.t. $f(x_n)\to\alpha$ (usually called minimizing sequence). As $(x_n)\subset[0,1]$, there is converging subsequence i.e. $x_{n_k}\to x$. But then, by continuity of f,

$$f(x)=\lim_{n_k\to\infty} f(x_{n_k}) = \alpha = \inf_{x\in[0,1]} f(x),\quad \text{as desired.}$$

Lower semicontinuity of f is sufficient here as

$$\underbrace{\inf_{x\in[0,1]} f(x)}_{=\,\alpha} \le f(x) \le \liminf_{n_k\to\infty} f(x_{n_k}) = \alpha \Rightarrow \underline{f(x)=\alpha}.$$

Compare this with Remark to the Problem W6. Note that similar technique can be used for $f: E\to\mathbb{R}$, convex and lower semicontinuous, assuming we can extract weakly converging subsequences.

W8 Let H be separable Hilbert space and let $\{e_k\}$ be its orthonormal Schauder basis such that $H=\overline{\mathrm{span}(e_1,e_2,\dots)}$.

Let $\{x_n\}\subset H$ be a bounded sequence. ~~[illegible]~~ By RRT each x_n defines an element of H^*, i.e. $x_n(h)=\langle x_n,h\rangle$ $\forall_{h\in H}$.

Moreover, $|\langle x_n,h\rangle| \le \|x_n\|\,\|h\| \le \left(\sup_{n\in\mathbb{N}}\|x_n\|\right)\|h\| < \infty$ by Assumption. Therefore;

For each i, we can choose subsequence $\langle x_{n_k}, e_i\rangle$ $\{x_{n_k}\}$ so that converges. In particular, proceeding diagonally, we can find subsequence $\{x_{n_k}\}\subset\{x_n\}$ s.t. $\langle x_{n_k}, e_i\rangle$ has limit for all $i\in\mathbb{N}$.

we find family of subsequences $\{x_n\}\supset\{x^{(1)}\}\supset\{x^{(2)}\}\supset\{x^{(3)}\}$

Note that this limit exists, by linearity, for all $y \in \mathrm{span}(e_1, e_2, \dots)$

For $y \in \mathrm{span}(e_1, e_2, \dots)$, let

$$\varphi(y) = \lim_{n_k \to \infty} (x_{n_k}, y) \quad (*)$$

Note that $\varphi \in (\mathrm{span}(e_1, e_2, \dots))^*$ as $|\varphi(y)| \leq \left(\sup_n \|x_n\|\right) \|y\|$.

As $\overline{\mathrm{span}(e_1, e_2, \dots)} = H$, φ has a unique extension to H (see Big Homework 1) and this extension preserves the norm. Call this extension $\tilde{\varphi}$. As $\tilde{\varphi} \in H^*$, by RRT, $\exists_{x \in H}$ $\tilde{\varphi}(h) = \langle x, h \rangle$.

Note that $(*)$ implies $\lim_{n_k \to \infty} \langle x_{n_k}, y) = (x, y)$ for all $y \in \mathrm{span}(e_1, e_2, \dots)$. But this limit also holds for all $y \in H = \overline{\mathrm{span}(e_1, e_2, \dots)}$. Indeed, let $\varepsilon > 0$. Fix $y \in H$. Find y_ε s.t. $\|y_\varepsilon - y\| \leq \varepsilon$ (by density) and $y_\varepsilon \in \mathrm{span}(e_1, e_2, \dots)$. Then

$$|(x_{n_k} - x, y)| \leq \underbrace{|(x_{n_k} - x, y - y_\varepsilon)|}_{\leq \|x_{n_k} - x\| \|y - y_\varepsilon\| \leq C\varepsilon,\ C = 2\sup_n \|x_n\|} + |(x_{n_k} - x, y_\varepsilon)|$$

$$\Rightarrow \limsup_{n_k \to \infty} |(x_{n_k} - x, y)| \leq C\varepsilon.$$

As ε is arbitrary, the conclusion follows.

W9 Let $\{e_j\}_{j \in \mathbb{N}}$ be orthonormal basis of H (it exists as H is sep.) By Bessel's inequality $\forall_x$ $\sum_{k \geq 1} \langle x, e_k \rangle^2 \leq \|x\|^2$ hence $\forall_x$ $\langle x, e_k \rangle \to 0$ as $k \to \infty$. It follows that $\{e_j\}_{j \in \mathbb{N}}$ is desired sequence