Functional Analysis (WS 19/20), Problem Set 7

(Radon - Nikodym theorem and space of signed measures)

In what follows, let X be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of X.

Signed measures and Hahn-Jordan decomposition theorem Let μ be a real-valued, signed measure on $(X,\mathcal{F})$. Then, there exists unique decomposition $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are nonnegative measures and μ^+, μ^- have disjoint supports.

Total variation measure and norm We define a nonnegative measure $|\mu| = \mu^+ + \mu^-$ called total variation measure. We also set $\|\mu\|_{TV} = |\mu|(X)$. The signed measure μ is said to be finite (or bounded) if $\|\mu\|_{TV} < \infty$.

Radon-Nikodym Theorem Let μ be σ-finite signed measure and ν be a σ-finite nonnegative measure. Suppose that ν is absolutely continuous with respect to μ, i.e. $\nu(A) = 0$ implies $|\mu|(A) = 0$ for all $A \in \mathcal{F}$ (we write $\mu << \nu$). Then, there exists uniquely determined (up to the null sets of ν) function such that

$$\mu(A) = \int_A f(x)\, d\nu(x).$$

We say that f is the derivative of μ with respect to ν. Abusing notation, we usually write $\frac{d\mu}{d\nu} = f$ or $d\mu = f d\nu$.

Radon-Nikodym Theorem

R1. ♣ (**absolute continuity**) Let μ, ν be nonnegative measures such that ν is bounded. Prove that $\nu << \mu$ if and only if for any $\epsilon > 0$ there is $\delta > 0$ such that $\nu(A) \le \epsilon$ whenever $\mu(A) \le \delta$.

R2. Problem R1 may fail if ν is not finite: on $(0,1)$ consider ν given with $d\nu = \frac{1}{x}dx$ and $d\mu = dx$ where dx is the Lebesgue measure. Alternatively, take ν to be the counting measure on $\mathbb{N}$ and $\mu(E) = \sum_{n\in E} 2^{-n}\delta_n$.

R3. The σ-finiteness assumption in Radon-Nikodym theorem is essential. Consider $X = [0,1]$. Let m be Lebesgue measure and μ be a counting measure on $[0,1]$. Prove that $m << \mu$ but there is no function f such that $dm = f\, d\mu$.

R4. Let $(X,\mathcal{F},\mu)$ be a σ-finite measure space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $f \in L^1(\mu)$ and ν be a restriction of μ to $\mathcal{G}$. Prove that there exists $g \in L^1(\nu)$ (in particular: g is $\mathcal{G}$-measurable) such that

$$\int_E f\, d\mu = \int_E g\, d\nu \qquad \forall_{E\in\mathcal{G}}.$$

Moreover, g is uniquely determined up to the null sets of ν.

R5. (**conditional expectation**) Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and X be a real-valued random variable such that $\mathbb{E}|X| < \infty$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Prove that there exists a $\mathcal{G}$-measurable random variable Y such that

$$\mathbb{E}X\mathbb{1}_A = \mathbb{E}Y\mathbb{1}_A \qquad \forall_{A\in\mathcal{G}}.$$

We usually write $Y = \mathbb{E}(X|\mathcal{G})$ and call Y a conditional expectation of X with respect to $\mathcal{G}$.

Spaces of measures

M1. Let $f \in L^1(\mu)$ and $\nu(A) = \int_A f d\mu$ where μ is a nonnegative measure. Find Hahn-Jordan decomposition of ν. Prove that $|\nu| = \int_A |f| d\mu$ and $\|\nu\|_{TV} = \|f\|_{L^1(\mu)}$.

M2. ♣ Prove that the set of real-valued signed and finite measures $\mathcal{M}(X)$ equipped with $\|\cdot\|_{TV}$ as a norm is a normed space.

M3. ♣ Prove that the set of real-valued signed and finite measures $\mathcal{M}(X)$ equipped with $\|\cdot\|_{TV}$ as a norm is a Banach space. *Hint*: A delightful application of Radon-Nikodym to the measure $\sum_{i=1}^{\infty} \frac{1}{2^n} |\mu_n|$.

M4. Let X be an uncountable set. Prove that the set of real-valued signed and finite measures $\mathcal{M}(X)$ equipped with $\|\cdot\|_{TV}$ as a norm is not separable.

M5. ♣ (**continuity pathology**) Compute $\|\delta_a - \delta_b\|_{TV}$. Deduce that if $x_n \to x$ and $x_n \neq x$ then it does not hold that $\delta_{x_n} \to \delta_x$ in the total variation norm. *Remark:* This shows that the total variation norm does not keep information about pointwise behaviour of measures.

M6. (**1-Wasserstein distance**) On the set of probability measures on $(X, \mathcal{F}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite first moment (i.e. $\int_X |x| d\mu(x) < \infty$) we define

$$W_1(\mu, \nu) = \sup \int_X f(x) \, d(\mu(x) - \nu(x)),$$

where the supremum is taken over all Lipschitz functions with Lipschitz constant at most 1.

- Verify that W_1 is indeed a metric on the space of probability measures.
- Prove that it cannot be extended to the set of finite and nonnegative measures $\mathcal{M}^+(X)$.
- Prove that if $x_n \to x$ then $W_1(\delta_{x_n}, \delta_x) \to 0$. Hence, continuity pathology does not occur here.

M7. (**flat metric**) Check that the following modification of Wasserstein distance:

$$p_F(\mu, \nu) = \sup \int_X f(x) \, d(\mu(x) - \nu(x)),$$

where the supremum is taken over all Lipschitz and bounded functions with Lipschitz constant and supremum norm at most 1, is well defined on the space of nonnegative finite measures.