Problem Set 7

Radon-Nikodym Theorem and spaces of measures

- Hahn-Jordan decomposition: $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are nonnegative measures (as in Analysis II). Moreover, $\exists_A \; \mu^+(X \setminus A) = 0, \mu^-(A) = 0$ and this set is uniquely determined: if A' satisfies also these conditions, $\mu(A \Delta A') = 0$.

- $|\mu| := \mu^+ + \mu^-$ = nonnegative measure

  $\|\mu\|_{TV} = |\mu|(X)$ = just a number, will be used as the norm.

- R-N thm: we want to find density of μ (signed measure) with respect to the nonnegative one v. Such density f satisfies

$$(*) \mu(A) = \int_A f \, dv$$

Hint: one can remember this condition from (*).

The necessary condition is $|\mu| \ll v$, i.e. absolute continuity.

R1 This exercise shows some equivalent condition for absolute continuity. We claim

$$v \ll \mu \iff \forall_{\varepsilon > 0} \exists_{\delta > 0} \; \mu(A) \leq \delta \Rightarrow v(A) \leq \varepsilon$$

($\Leftarrow$) Easy. Let A be such that $\mu(A) = 0$. Then $v(A) \leq \varepsilon \; \forall_{\varepsilon > 0} \Rightarrow v(A) = 0$.

(⇒) Aiming at contradiction suppose $\exists_{\varepsilon>0} \forall_{\delta>0} \exists_{A_\delta}$ $\mu(A_\delta) \leq \delta$ and $\nu(A) \geq \varepsilon$.

We take $\delta = 2^{-n}$ and we get family of sets A_n s.t. $\mu(A_n) \leq 2^{-n}$ but $\nu(A_n) \geq \varepsilon$. Consider $A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Clearly, if ν is finite, we can use lower continuity of measures to get

$A_1 \supset A_2 \supset \dots$ and $\nu(A_1) < \infty \Rightarrow$

$\Rightarrow \nu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \nu(A_i)$

But: this does not hold if $\nu(A_1) = \infty$ (for inst, think about $A_k = [k,\infty) \subset \mathbb{R}$).

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \varepsilon.$$

On the other hand $\mu(A) \leq \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} 2^{-k}$ $\forall_n \Rightarrow \mu(A) = 0$.

Contradiction with $\nu \ll \mu$. $\square$.

R2 $\nu(A) = \int_A \frac{1}{x}\,dx$ $\mu(A) = \int_A 1\,dx$. $\Rightarrow \nu \ll \mu$

In particular, $\frac{1}{x}$ is density of ν with respect to μ. We claim that $\varepsilon-\delta$ condition does not hold. Indeed

$$\mu([0,a]) = a$$
$$\nu([0,a]) = \int_0^a \frac{1}{x} = \infty$$

$\Rightarrow \forall_{\varepsilon>0} \exists_{\delta>0}$ $\mu(A) < \delta \Rightarrow \nu(A) \leq \varepsilon$. does not hold.

Similar example with ν = counting measure and $\mu(A) = \sum_{n\in E} 2^{-n} \delta_n$

Indeed, let $A_k = \{k, k+1, \dots\}$. Then, $\nu(A_k) = \infty$ but $\mu(A_k) \to 0$ as $k\to\infty$ as $\left(\sum_{n=k}^{\infty} 2^{-n} \to 0 \text{ as } k \to \infty\right)$.

(R3) σ-finiteness assumption is essential in R-N theorem.

$X = [0,1]$, μ = counting measure on $[0,1]$, m = Leb. measure.

<u>Claim</u>: $m \ll \mu$. Indeed, let A be s.t. $\mu(A) = 0$. Then A has to be empty so indeed $m(A) = 0$.

<u>Claim</u>: m has no density wrt μ. Suppose that $m(A) = \underbrace{\int_A f \, d\mu}_{\text{this is integral in Lebesgue sense}}$ for some $f : [0,1] \longrightarrow [0,\infty]$.

$$0 = m(\{x\}) = \int_{\{x\}} f \, d\mu = f(x)\mu(\{x\}) \Rightarrow \forall_{x \in [0,1]} \; f(x) = 0 \text{ or } \mu(\{x\}) = 0$$

~~But then $m([0,1]) = \sup_{\substack{0 \le s \le f \\ s \text{ simple}}} \{ \int_A s \, d\mu \} = 0 \Rightarrow$ contradiction.~~ ~~$\sum_{k=1}^{m} 1\!\!1_{A_k} a_k$~~

But μ is counting measure so

$$\mu(\{x\}) = 1 \Rightarrow f(x) = 0 \text{ for all } x \in [0,1] \Rightarrow m([0,1]) = \int_0^1 f \, d\mu = 0$$

$\Rightarrow$ contradiction.

M1 $f \in L^1(\mu)$, $\nu(A) = \int_A f\,d\mu$, μ is nonnegative

$f = f^+ - f^-$, $f^+, f^- \geq 0$

f^+: positive part of f; f^-: negative part of f

Then $\nu(A) = \int_A f\,d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \int |f|\,d\mu.$

Note that $\nu \ll \mu$ so the null sets of μ are in particular the null sets of ν and so we have found Jordan decomposition of ν: (ν^+, ν^-). It is unique up to null sets of ν.

Total variation: $|\nu|(A) = \int_A f^+ d\mu + \int_A f^- d\mu = \int_A |f|\,d\mu \Rightarrow$

$\|\nu\|_{TV} = \|f\|_{L^1(\mu)}.$

M2 $\mathcal{M}(X)$ is a vector space: here we use finiteness assumption (otherwise we would have to deal with expressions $\infty - \infty$).

1) $\|\nu\|_{TV} = 0 \iff \nu^+(X), \nu^-(X) = 0 \iff \nu = 0$

2) $\|t\nu\|_{TV} = t\nu^+(X) + t\nu^-(X) = t\|\nu\|_{TV}$

3) $\|\nu+\mu\|_{TV} \leq \|\nu\|_{TV} + \|\mu\|_{TV}$

$\nu+\mu \in \mathcal{M}(X) \Rightarrow (\nu+\mu) = (\nu+\mu)^+ - (\nu+\mu)^-$. Let P be the positive set and N be the negative set ($P \cup N = X$, $P \cap N = \emptyset$).

$|\nu+\mu|(X) = (\nu+\mu)^+(X) + (\nu+\mu)^-(X)$

$(\nu+\mu)^+(X) = (\nu+\mu)(X \cap P) \leq (\nu^+ + \mu^+)(X \cap P) \leq \underbrace{(\nu^+ + \mu^+)(X)}_{= \nu^+(X) + \mu^+(X)}$

$\leq \nu(\ldots) + \mu(\ldots)$

$(\nu+\mu)^-(X) = -(\nu+\mu)(X \cap N) \leq (\nu^- + \mu^-)(X \cap N) \leq \nu^-(X) + \mu^-(X)$

(M3) Nice application of Radon–Nikodym Theorem. Let $\{\mu_n\}_n$ be a Cauchy sequence wrt $\|\cdot\|_{TV}$. Let $v = \sum_{n=1}^{\infty} |\mu_n| \frac{1}{2^n}$ be a finite (it is finite as μ_n is bdd in TV as it is Cauchy), nonneg. measure

$\forall_n \ \mu_n << v \implies \exists_{f_n} \ \mu_n(A) = \int f_n \, dv$. Moreover, as μ_n is a finite measure, $f_n \in L^1(v)$. By M1

$$\|\mu_n - \mu_m\|_{TV} = \|f_n - f_m\|_{L^1(v)} \quad \text{so } \{f_n\} \text{ is Cauchy in } L^1(v).$$

In particular $f_n \xrightarrow{L^1(v)} f$ for some $f \in L^1(v)$. It follows that for $d\mu = f\,dv$ we have

measure with density f

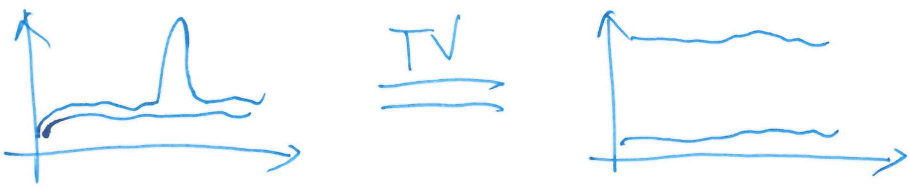
$$\|\mu_n - \mu\|_{TV} = \|f_n - f\|_{L^1} \longrightarrow 0 \implies \mu_n \to \mu \text{ in TV}$$

as desired. $\square$

(M4) For each $x \in X$ consider δ_x. Then $\|\delta_x - \delta_y\|_{TV} = 2$ for $x \neq y$. Since $\{\delta_x\}$ is uncountable, it follows that $(\mathcal{M}(X), \|\cdot\|_{TV})$ is not separable.

(M5) By M4: $\|\delta_a - \delta_b\|_{TV} = \begin{cases} 2 & a \neq b \\ 0 & a = b \end{cases}$. In particular, if $x_n \to x$

then $\|\delta_{x_n} - \delta_x\|_{TV} = 2 \ \forall_n$ assuming $x_n \neq x$. This shows that topology on $\mathcal{M}(X)$ is ~~too~~ strong for applications (in particular: it does not see local properties of measures). For example:



M6

$$W_1(\mu,\nu) = \sup_{\substack{f \text{ Lipschitz} \\ |f|_{Lip} \leq 1}} \int_{\mathbb{R}^d} f(x)\, d(\mu(x) - \nu(x))$$

missing info in the first version of PS:
→ measures with integrable first moment

(a) • $W_1(\mu,\nu) = \sup \int (f(x) - f(0))\, d(\mu(x) - \nu(x)) \leq$

$$\leq \sup \int |f(x) - f(0)|\, |\mu - \nu|(x) \leq \underbrace{\int |x|\, d\mu(x)}_{} + \underbrace{\int |x|\, d\nu(x)}_{} < \infty.$$

finite by assumption.

• Clearly, $W_1(\mu,\nu) = W_1(\nu,\mu)$ as if f is 1-Lipschitz then $-f$ is also 1-Lipschitz.

• Triangle inequality: $\int f(x)\, d(\mu_1(x) - \mu_2(x)) =$

$$= \int f(x)\, d(\mu_1(x) - \mu_3(x)) + \int f(x)\, d(\mu_3(x) - \mu_2(x)) \leq W_1(\mu_1,\mu_3) + W_1(\mu_3,\mu_2)$$

Take sup on the (LHS) to deduce $W_1(\mu_1,\mu_2) \leq W_1(\mu_1,\mu_3) + W_1(\mu_3,\mu_2)$.

• If $\mu = \nu$ then $W_1(\mu,\nu) = 0$. Convercely, suppose that $W_1(\mu,\nu) = 0$. We claim $\mu = \nu$. For if not, there is a (closed) cube $Q \subset \mathbb{R}^d$ such that $\mu(Q) \neq \nu(Q)$.

→ Fix $\varepsilon > 0$. Let $f_\varepsilon(x) = \begin{cases} 1 & x \in Q \\ -\frac{\operatorname{dist}(x,Q)}{\varepsilon} + 1 & \operatorname{dist}(x,Q) \leq \varepsilon \\ 0 & \operatorname{dist}(x,Q) > \varepsilon \end{cases} =$

$= \max\left(0, 1 - \frac{\operatorname{dist}(x,Q)}{\varepsilon}\right)$. This is Lipschitz function with Lip. const. estimated by $\frac{1}{\varepsilon}$. Moreover $f_\varepsilon(x) \longrightarrow \mathbb{1}_Q$.

→ As $W_1(\mu,\nu) = 0 \Rightarrow \forall_{f \in Lip_1} \int f\, d\mu = \int f\, d\nu \Rightarrow \forall_{f \in Lip} \int f\, d\mu = \int f\, d\nu$ (by scaling).

Therefore, we get $\forall_{\varepsilon>0}$ $\int f_\varepsilon \, d\mu = \int f_\varepsilon \, d\nu$. By DCT we obtain

$$\int \mathbb{1}_Q \, d\mu = \int \mathbb{1}_Q \, d\nu \Rightarrow \mu(Q) = \nu(Q) \Rightarrow \text{contradiction}$$

(b) W_1 cannot be extended: Let $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\mu(\mathbb{R}^d) \neq \nu(\mathbb{R}^d)$.

Consider $f = c \in \mathbb{R}$ ← Lipschitz function. Then $\forall_{c \in \mathbb{R}}$

$$W_1(\mu,\nu) \geq \int f \, d(\mu-\nu) = c\,|\mu(\mathbb{R}^d) - \nu(\mathbb{R}^d)| \neq 0$$

As c is arbitrary, $W_1(\mu,\nu) = \infty$.

(c) Let $x_n \longrightarrow x$. Then $W_1(\delta_{x_n}, \delta_x) = \sup\limits_{f \in Lip_1} \int f\,(d\delta_{x_n} - d\delta_x) =$

$= \sup\limits_{f \in Lip_1} f(x_n) - f(x) \leq |x_n - x| \longrightarrow 0$. Hence, continuity pathology does not occur here, i.e. Wasserstein metric carries more information then total variation.

M7 Flat metric:

$$\rho_F(\mu,\nu) = \sup_{\|f\|_\infty \leq 1,\ |f|_{Lip} \leq 1} \int f\,(d\mu - d\nu).$$

We have $\rho_F(\mu,\nu) \leq \sup\left[\int |f|\, d|\mu| + \int |f|\, d|\nu|\right] \leq$

$\leq \|\mu\| + \|\nu\| < \infty$.

In practical applications, flat metric is used to study flows on the space of measures:

→ no continuity pathology (so carries enough information)

→ can describe non-conservative problems (different total mass).