Problem Set 8

(spectrum and adjoints on HS)

Hilbert spaces with scalar field $\mathbb{C}$: like in $\mathbb{R}$ but the scalar product satisfies:

(1) linearity in first argument $\rightsquigarrow$ in the second argument we take conjugate by (2)

(2) $\langle x, y\rangle = \langle y, x\rangle$

(3) (or positive definiteness $\cdot\ \langle x,x\rangle > 0\ x\neq 0$, $\cdot\ \langle x,x\rangle = 0\ x=0$)

Example: $L^2(0,1)$: $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$. Then, we check (2):

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx = \overline{\int_0^1 \overline{f(x)}\, g(x)\,dy} = \overline{\int_0^1 g(x)\,\overline{f(x)}\,dx} = \overline{\langle g, f\rangle}$$

R1 T^* exists and is unique: we have to find T^* s.t. $\langle Tx, y\rangle = \langle x, T^* y\rangle$

For any y, let $\varphi_y(x) = \langle Tx, y\rangle$. We have $|\varphi_y(x)| \leq \|T\|\,\|x\|\,\|y\|$

so $\varphi_y \in H^*$. By RRT, $\exists!_{\tilde{y}}\ \varphi_y(x) = \langle x, \tilde{y}\rangle$. We set $T^* y = \tilde{y}$.

i.e. $\langle Tx, y\rangle = \langle x, T^* y\rangle$ as desired.

R3 $\langle Tx, y\rangle = \langle x, T^* y\rangle = \overline{\langle T^* y, x\rangle} = \overline{\langle y, T^{**}x\rangle} = \langle T^{**}x, y\rangle$

$$\Rightarrow \langle (T - T^{**})x, y\rangle = 0 \ \overset{\forall y}{\Rightarrow}\ T = T^{**}.$$

R4

For T_1: $\langle T_1 x, y\rangle = \langle x, T_1^* y\rangle$

For T_2: $\langle T_2 x, y\rangle = \langle x, T_2^* y\rangle$

For (T_1+T_2): $\langle (T_1+T_2)x, y\rangle = \langle x, (T_1+T_2)^* y\rangle$

But $\langle T_1 x, y\rangle + \langle T_2 x, y\rangle = \langle (T_1+T_2)x, y\rangle$ [linearity in the 1st argument]

Also $\langle x, T_1^* y\rangle + \langle x, T_2^* y\rangle = \langle x, T_1^* y + T_2^* y\rangle =$

$= \langle x, (T_1^* + T_2^*) y\rangle$ As adjoint is unique, the cond. follows. $\square$

<u>Note</u>: on complex Hilbert spaces, $\langle \cdot, \cdot \rangle$ is "conjugate linear" in the second argument, in the sense that

$$\langle x, a y_1 + b y_2\rangle = \bar{a}\langle x, y_1\rangle + \bar{b}\langle x, y_2\rangle .$$

R5

For T: $\langle Tx, y\rangle = \langle x, T^* y\rangle \quad / \cdot \lambda$

$\langle (\lambda T)x, y\rangle = \langle x, (\bar{\lambda} T^*) y\rangle$

For (λT): $\langle \lambda T x, y\rangle = \langle x, (\lambda T)^* y\rangle$

By uniqueness, $(\lambda T)^* = \bar{\lambda} T^*$. $\square$.

~~R6 First, T^* is invertible and bounded.~~

- ~~T^* is injective. Suppose $\exists_{y \in H}$ $T^* y = 0$. Then $\langle Tx, y\rangle = 0$ $\forall_{x\in H}$. As T is invertible $\exists_{x = T^{-1}y}$ $\Rightarrow$ $y = 0$.~~
- ~~T^* is surjective.~~

R7

$$\langle T_1 T_2 x, y\rangle = \langle T_2 x, T_1^* y\rangle = \langle x, T_2^* T_1^* y\rangle$$
$$\| $$
$$\langle x, (T_1 T_2)^* y\rangle$$

$\Rightarrow$ by uniqueness of adjoints $(T_1 T_2)^* = T_2^* T_1^*$.

R6 As T is invertible, T has left and right inverse, i.e. $TT^{-1} = I$, $T^{-1}T = I$. We apply R7 to get

(observe that $(I)^* = I$)

$$\begin{cases} (T^{-1})^* T^* = I \\ (T)^* (T^{-1})^* = I \end{cases} \Rightarrow T^* \text{ has left and right inverse}$$

$\Rightarrow T^*$ is invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

R8 $\ker T^* = (\operatorname{im} T)^\perp$ $(*)$

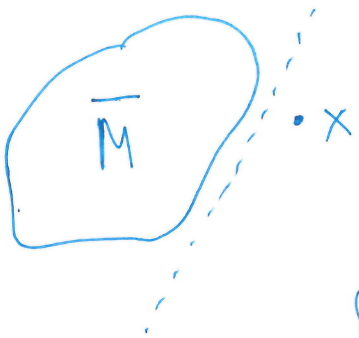
<u>Proof</u>: $x \in \ker T^* \iff T^* x = 0 \iff \forall_{y \in H} \ \langle y, T^* x \rangle = 0$

$\iff \langle Ty, x \rangle = 0 \ \forall_{y \in H} \iff x \in (\operatorname{im} T)^\perp$. $\square$.

Using $(*)$, we get $(\ker T^*)^\perp = ((\operatorname{im} T)^\perp)^\perp$. And we have to check that $((\operatorname{im} T)^\perp)^\perp = \overline{\operatorname{im} T}$.

We prove that if $M \subset H$ (linear subspace) $\Rightarrow \overline{M} = (M^\perp)^\perp$. We prove two implusions.

- $x \in M \Rightarrow x \in (M^\perp)^\perp$. Since $(M^\perp)^\perp$ is closed, $\overline{M} \subset (M^\perp)^\perp$.
- $x \in (M^\perp)^\perp$ and suppose that $x \notin \overline{M}$. There is a functional vanishing on $\overline{M}$ s.t. $\varphi(x) \neq 0$.



By RRT $\varphi(y) = \langle h, y \rangle$ for some $h \in H$.

As $\varphi|_{\overline{M}} = 0 \Rightarrow h \in M^\perp$. As $x \in (M^\perp)^\perp$

$\varphi(x) = \langle h, x \rangle = 0 \Rightarrow$ contradiction.

This concludes the proof.

(M1) $A : \mathbb{R}^n \to \mathbb{R}^n$ $\quad A \in \mathbb{C}^{n\times n}$. Scalar product $\langle x, y\rangle = x^T \bar{y}$.

$$\langle Ax, y\rangle = (Ax)^T \cdot \bar{y} =$$

$$= x^T A^T \bar{y} = x^T \cdot \left(\overline{\bar{A}^T y}\right) = \langle x, \bar{A}^T y\rangle \Rightarrow A^* = \bar{A}^T$$

(By uniqueness of adjoints...)

(M2) $H = \ell^2(\mathbb{Z})$, $x = (\dots, x_{-1}, x_0, x_1, \dots)$, $R =$ right shift

Clearly, $\|R\| = 1$, $R^{-1} =$ left shift.

$$\langle Rx, y\rangle = \sum_{k\in\mathbb{Z}} (Rx)_k \cdot y_k = \sum_{k\in\mathbb{Z}} x_{k-1} y_k = \langle x, \underset{\text{left shift}}{R^{-1}} y\rangle$$

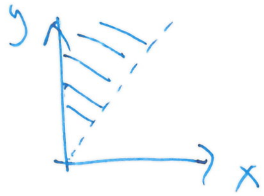
So it follows, $R^* = R^{-1}$.

(M3) $K : L^2(0,1) \to L^2(0,1)$ $\quad Kf(x) = \int_0^x f(y)\,dy$

$$|Kf(x)| \le \int_0^1 |f| \le \left(\int_0^1 |f|^2\right)^{1/2} = \|f\|_{L^2}$$

$\|Kf\|_{L^2} \le \|f\|_{L^2}$. $\Rightarrow$ K is bounded on $L^2(0,1)$.

$$\langle Kf, g\rangle = \int_0^1 Kf(x)\bar{g}(x)\,dx = \int_0^1 \int_0^x f(y)\bar{g}(x)\,dy\,dx$$



$$\underset{\text{Fubini}}{=} \int_0^1 \int_y^1 f(y)\bar{g}(x)\,dx\,dy =$$

$$= \int_0^1 f(y) \overline{\int_y^1 g(x)\,dx}\,dy = \langle f, K^* g\rangle$$

for $K^* g = \int_y^1 g(x)\,dx$.

M4 Note that $H = M \oplus M^\perp$. Let $x = \underset{\in M}{x_1} + \underset{\in M^\perp}{x_2}$, $y = \underset{\in M}{y_1} + \underset{\in M^\perp}{y_2}$.

Then
$$\langle P_M x, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \underbrace{\langle x_1, y_2\rangle}_{=0} = \langle x_1, y_1\rangle =$$
$$\langle x_1, y_1\rangle + \underbrace{\langle x_2, y_1\rangle}_{=0} = \langle x, y_1\rangle = \langle x, P_M y\rangle. \qquad \square.$$

M5 $A: H \to H$ bdd operator

justified by strong convergence of the series.

$$\langle e^A x, y\rangle = \left\langle \sum_{k\geq 0} \frac{A^k x}{k!}, y\right\rangle = \sum_{k\geq 0} \left\langle \frac{A^k x}{k!}, y\right\rangle =$$
$$= \sum_{k\geq 0} \left\langle x, \frac{(A^k)^* y}{k!}\right\rangle = \langle x, e^{A^*} y\rangle \Rightarrow \left(e^A\right)^* = e^{A^*}.$$

$$(A^k)^* = (A^*)^k$$

R7

M6 $T: L^2(0,1) \to L^2(0,1)$. $Tf(x) = \int_0^1 k(x,y) f(y)\, dy$

$$\langle Tf, g\rangle = \int_0^1 Tf(x)\overline{g(x)}\, dx = \int_0^1 \int_0^1 k(x,y) f(y) \overline{g(x)}\, dx =$$
$$= \int_0^1 \int_0^1 f(y) k(x,y) \overline{g}(x)\, dx\, dy = \int_0^1 f(y) \int_0^1 k(x,y)\overline{g}(x)\, dx\, dy$$
$$= \int_0^1 f(y) \overline{\int_0^1 \overline{k(x,y)}\, g(x)\, dx}\, dy = \langle f, T^* g\rangle$$

$$T^* g(y) = \int_0^1 \overline{k(x,y)}\, g(x)\, dx$$

Spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ does not have a bounded inverse$\}$.

Note that as T is assumed to be bounded, ↑ is equivalent to does not have an inverse.

S1 This is decomposition of the spectrum so that we can find out "what can go wrong".

- $A - \lambda I$ is not injective ← point
- $A - \lambda I$ is injective but not surj and image of $A - \lambda I$ is dense in H ← continuous
- $A - \lambda I$ is injective but not surj and image of $A - \lambda I$ is not dense ← residual

Note that these 3 cases excludes themselves. Also note that if λ does not satisfy any of the three cases above then all 3 cases below holds:

- $A - \lambda I$ is injective
- $A - \lambda I$ is surjective <u>or</u> image $A - \lambda I$ is not dense in H
- $A - \lambda I$ is surjective <u>or</u> image $A - \lambda I$ is dense in H.

$\Rightarrow A - \lambda I$ is bijective $\Rightarrow \lambda \notin \sigma(A)$. This shows desired decomposition.

S2 $\sigma(A) \in \overline{B(0, \|A\|)} \subset \mathbb{C}$. This follows as we know that

↑ closed ball!

$I - T$ is invertible if $\|T\| < 1$. Hence $T - \lambda I = \lambda\left(\frac{T}{\lambda} - I\right) = -\lambda\left(I - \frac{1}{\lambda}T\right)$. If $|\lambda| > \|T\|$, this operator is invertible.

$\square$.

(S3) $\rho(A)$ is open subset of $\mathbb{C}$ (lecture)

Hence $\sigma(A)$ is closed subset of $\mathbb{C}$ and bounded by (S2).

It follows that $\sigma(A)$ is compact.

(S4) This result somehow relates to the classical linear algebra fact that every matrix has an eigenvalue over $\mathbb{C}$. The proof is based on Complex Analysis and is announced as <u>Special Problem</u>.

This result <u>DO NOT</u> hold over $\mathbb{R}$.

(S5) We're gonna prove sth stronger. Namely, let p be a polynomial. We claim that

$$\boxed{\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.}$$

<u>updated</u>
problem set

Let $\lambda \in \mathbb{C}$ and $q(z) = -\lambda + p(z)$. As q is a polynomial over $\mathbb{C}$, we have that

$$q(z) = c(z-\lambda_1)(z-\lambda_2)\dots(z-\lambda_n) \quad \exists_{\lambda_1,\dots,\lambda_n \in \mathbb{C}}.$$

Then, $\lambda \notin \sigma(p(T)) \iff p(T) - \lambda I$ is invertible $\iff$

$\iff q(T)$ is invertible $\iff$

$\iff (T-\lambda_1 I)(T-\lambda_2 I)\dots(T-\lambda_n I)$ is invertible $\overset{(*)}{\iff}$

$\iff \forall_{k=1,\dots,n}\ T-\lambda_k I$ is invertible $\iff$

$\iff \forall_{k=1,\dots,n}\ \lambda_k \notin \sigma(T) \iff \forall_{\mu\in\sigma(T)}\ q(\mu) \neq 0$

$\iff p(\mu) \neq \lambda \quad \forall_{\mu \in \sigma(T)}$. We have to justify (*).

Proof of (*):

Note that $(T-\lambda I)(T-\mu I) = (T-\mu I)(T-\lambda I)$ $\forall_{\mu,\lambda \in \mathbb{C}}$.
(i.e. commutation property)

($\Rightarrow$) trivial

($\Leftarrow$) First, $(T-\lambda_1 I)$ has to be surjective and $(T-\lambda_n I)$ has to be injective. Using commutation property, $(T-\lambda_i I)$ has to be injective and surjective for all $i=1,\dots,n$.

$\square$.

Remark: This is the way to solve problems like "what is the spectrum of T^2-3T+2 ?".

(S6) As $A-\lambda I$ is a map between finite dimensional spaces, it is injective ~~and~~ if and only if surjective (this is "rank-nullity theorem: $\dim\ker(A-\lambda I) + \dim \operatorname{im}(A-\lambda I) = n$). In particular, $A-\lambda I$ cannot be both injective and not surjective. The conclusion follows.

(S7) Such problems are solved as follows: we start with determining eigenvalues and we try to see for which $\lambda \in \mathbb{C}$ operator $M-\lambda I$ can be inverted.

point spectrum (eigenvalues)

- point spectrum: suppose $\exists_{\lambda\in\mathbb{C}} \exists_f (M-\lambda I)f = 0 \Rightarrow$
$x f(x) - \lambda f(x) = 0$ for a.e. $x \in [0,1] \Rightarrow x=\lambda$ or $f(x)=0$ for a.e. $x\in[0,1] \Rightarrow f(x)=0$ for a.e. $x\in[0,1] \Rightarrow$ no eigenvalues.

• try to invert operator: the natural inverse of $M-\lambda I$ is

$$N_\lambda f(x) = \frac{1}{x-\lambda} f(x).$$

Clearly, if $\lambda \notin [0,1]$, then $\inf_{x\in[0,1]} |x-\lambda| > 0$ (for instance, because $[0,1]$ is closed in $\mathbb{C}$). Call $c_\lambda := \inf_{x\in[0,1]} |x-\lambda| > 0$. Then $N_\lambda : L^2(0,1) \to L^2(0,1)$ is a bdd operator. Indeed,

$$\int_0^1 \left|\frac{1}{x-\lambda} f\right|^2 \leqslant \frac{1}{c_\lambda^2} \int_0^1 |f|^2 \Rightarrow \|N_\lambda\|_{L^2} \leq \frac{1}{c_\lambda} \|f\|_{L^2}.$$

This proves $\sigma(M) \subset [0,1]$.

After these two steps, there is room for some creativity. Actually, we have seen that with $\lambda \in [0,1]$ the main trouble is with term $\frac{1}{x-\lambda}$. Hence, we may hope to construct approximation by truncating this term.

<u>Claim</u>: $\sigma(M) = [0,1]$ and $\sigma(M)$ is purely continuous.

<u>Proof</u>: Clearly, if $\lambda \in [0,1]$ then $M-\lambda I$ is injective. It is not surjective for if not, $1 \in L^2(0,1)$ and $1 = f(x)(x-\lambda)$

$$\Rightarrow f(x) = \frac{1}{x-\lambda} \notin L^2(0,1) \Rightarrow \text{contradiction}.$$

Finally, we prove that image of $M-\lambda I$ is dense in $L^2(0,1)$. Fix $f \in L^2(0,1)$ and $\lambda \in \mathbb{C}$ and consider (as suggested...)

$$f_n(x) = \begin{cases} f & |x-\lambda| \geq \frac{1}{n} \\ 0 & |x-\lambda| < \frac{1}{n} \end{cases} \qquad f_n(x) \to f(x) \text{ in } L^2(0,1) \text{ by DCT}$$

Moreover, $f_n(x)$ is in the image of $M - \lambda I$ because

$$f_n(x) = (M - \lambda I) g_n = (x-\lambda) g_n(x) \Rightarrow g_n(x) = \frac{f_n(x)}{x-\lambda}$$

$$\Rightarrow g_n(x) = \frac{f_n(x)}{x-\lambda} \mathbb{1}_{\{|x-\lambda| \geq \frac{1}{n}\}} \quad \text{as } f_n = 0 \text{ on } \{|x-\lambda| < \tfrac{1}{n}\}.$$

So $f_n = (M-\lambda I) g_n$. Hence, as $f_n \to f$ in $L^2(0,1)$, image of $M - \lambda I$ is dense in H. $\square$.

S8 • Again, we start with point spectrum. Suppose $\exists_\lambda \exists_{x \neq 0}$ $(A-\lambda I)x = 0$

$$\Leftrightarrow \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \dots\right) - \lambda (x_1, x_2, x_3, \dots) = \left(0 - \lambda x_1, \frac{x_1}{1} - \lambda x_2, \dots, \frac{x_k}{k} - \lambda x_{k+1}, \dots\right) = 0.$$

If $\lambda = 0$ then $x_1 = x_2 = \dots = 0$. Suppose $\lambda \neq 0$. Then $x_1 = 0$ and we proceed term by term to get again $x_1 = x_2 = \dots = 0$. Hence, no eigenvalues and $A - \lambda I$ is always injective.

• We try to find "natural inverse". $(A - \lambda I)x = \left(0 - \lambda x_1, \frac{x_1}{1} - \lambda x_2, \dots, \frac{x_k}{k} - \lambda x_{k+1}, \dots\right)$

Suppose $(A - \lambda I)x = y$ for some $y \in \ell^2$. Can we find x? We have

$$\begin{cases} -\lambda x_1 = y_1 \\ \frac{x_k}{k} - \lambda x_{k+1} = y_{k+1} \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{y_1}{\lambda} \\ x_{k+1} = \left(\frac{x_k}{k} - y_{k+1}\right)\frac{1}{\lambda} \end{cases} \text{ if } \lambda \neq 0.$$

We need to find whether $x \in \ell^2$? Note that $(a+b)^2 \leq 2a^2 + 2b^2$ so

$$\sum_{k=m}^{n} |x_{k+1}|^2 \leq 2 \sum_{k=m}^{n} \left|\frac{x_k}{k\lambda}\right|^2 + \underbrace{2 \sum_{k=m}^{n} \left|\frac{y_{k+1}}{\lambda}\right|^2}_{\leq \frac{2}{\lambda^2} \sum_{k \geq 1} |y_k|^2 \text{ as } y \in \ell^2}$$

Let N be such that $1 - \frac{2}{N^2\lambda^2} \geq \frac{1}{2}$. Using bound above:

$$\sum_{k=N}^{n} |x_{k+1}|^2 \leq \frac{2}{N^2\lambda^2} \sum_{k=N}^{n} |x_k|^2 + \frac{2}{\lambda^2} \sum_{k\geq 1} |y_k|^2$$

$$\Rightarrow \underbrace{\left(1 - \frac{2}{N^2\lambda^2}\right)}_{\geq \frac{1}{2}} \sum_{k=N}^{n} |x_{k+1}|^2 \leq \frac{2}{N^2\lambda^2} |x_N|^2 + \frac{2}{\lambda^2} \sum_{k\geq 1} |y_k|^2$$

$$\Rightarrow \frac{1}{2} \sum_{k=N}^{n} |x_{k+1}|^2 \leq C(N, x_N, \lambda, y) \Leftarrow \text{bound independent of } n \geq N.$$

$$\Rightarrow \sum_{k=N}^{\infty} |x_{k+1}|^2 \text{ is finite} \Rightarrow \sum_{k\geq 1} |x_k|^2 \text{ is finite.}$$

- We are left to check $\lambda = 0$. But here $A - \lambda I = (0, x_1, \frac{x_2}{2}, \dots)$. We see that $A - \lambda I$ cannot be surjective (there are sequences in ℓ^2 with nonzero first component!). Moreover, its image is clearly not dense

Hence, A has purely residual spectrum and $\sigma(A) = \{0\}$.

S9 Approximate spectrum: $\lambda \in \mathbb{C}$ s.t. $\exists_{x_n, \|x_n\| = 1}$ $(A - \lambda I)x_n \to 0$.

- If λ is in app. spectr, then $\lambda \in \sigma(A)$: suppose $A - \lambda I$ is invertible (i.e. $\lambda \notin \sigma(A)$), then by Inverse Mapping Theorem, there is a constant C s.t. $\|x\| \leq C \|(A - \lambda I)x\|$. Apply this to the seq. x_n to get a contradiction.

- point spectrum is in approx. spectrum: indeed, we take a constant sequence consisting of eigenvector.

- continuous spectrum is in approximate spectrum: let $\lambda \in \sigma_c(A)$ and suppose $\lambda \notin \sigma_p(A)$ (we studied this case above). Then $A-\lambda I$ is injective and we can define its inverse $(A-\lambda I)^{-1}: \mathrm{im}(A-\lambda I) \to H$. We know from $\lambda \in \sigma_c(A)$ that $\mathrm{im}(A-\lambda I)$ is dense in H.

Suppose that $\lambda \notin \sigma_{app}(A)$. Then $\inf\limits_{\|x\|=1} \|(A-\lambda I)x\| > 0$. By linearity there is a constant C s.t. $\forall_x \ \|(A-\lambda I)x\| \geq C\|x\|$. Hence, $(A-\lambda I)^{-1}: \mathrm{im}(A-\lambda I) \to H$ is a bounded operator. As H is ~~Hilbert~~ (in particular, it is a Banach space), $(A-\lambda I)^{-1}$ has a unique bounded extension up to H. This extension is inverse of $A-\lambda I$:

- $(A-\lambda I)^{-1}(A-\lambda I) = I$ as here it acts on the image
- $(A-\lambda I)(A-\lambda I)^{-1} = I$: this holds if $x \in \mathrm{im}(A-\lambda I)$.

If $x \in H$, let $\{x_n\} \subset \mathrm{im}(A-\lambda I)$, $x_n \to x$. We have

$$(A-\lambda I)(A-\lambda I)^{-1}x_n = x_n.$$ Using boundedness we get conclusion.

But: this is contradiction with $\lambda \in \sigma(A)$.

Conclusion: $\sigma_{app}(A) \subset \sigma(A)$, $\sigma_p(A) \subset \sigma_{app}(A)$, $\sigma_c(A) \subset \sigma_{app}(A)$.

S10 (a) Actually, operator from S8 has the property that $\sigma_r(A) = \sigma_{app}(A) = \{0\}$. Indeed, if $e_j = (0, 0, \ldots, 0, \underset{\uparrow j}{1}, 0, \ldots)$

then $(A - 0I)e_j = \frac{e_{j+1}}{j+1} \to 0$ in ℓ^2 but $\|e_j\| = 1$.

(b) $\rightsquigarrow$ Special Problem announced. I don't know the answer :)

S11 Let $\{k_j\}_{j=1}^{\infty}$ be a countable dense subset of $K \subset \mathbb{C}$ and $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(0,1)$. We define an operator

$$Te_j = k_j e_j \qquad \forall_j$$

T is well-defined on $\operatorname{span}\{e_1, e_2, \dots\}$ = finite linear comb. of e_i.

We set $Tu = \sum_{j \geq 1} k_j \langle u, e_j \rangle e_j$. This series is convergent in H.

Proof of series convergence

Indeed, let $S_m = \sum_{j=1}^{m} k_j \langle u, e_j \rangle e_j$. We check that $\{S_m\}_m$ is a Cauchy sequence. Indeed, $\|S_m - S_n\|^2 = \sum_{j=n+1}^{m} k_j^2 \langle u, e_j \rangle^2 \leq$

$$\leq \left(\sup_j |k_j|^2\right) \cdot \underbrace{\sum_{j=n+1}^{m} \langle u, e_j \rangle^2}_{} \quad (*)$$

this series is convergent due to Bessel's inequality □

Moreover, T given by $Tu = \sum_{j \geq 1} k_j \langle u, e_j \rangle e_j$ defines a bounded operator on H: by Banach-Steinhaus sequence of bounded operators convergent pointwisely defines a bounded operator!

→ here, B-S is probably too much but some argument is important! In fact, we can just use uniform bound (*).

Clearly, by construction, $k_j \in \sigma(T) \Rightarrow K \subset \sigma(T)$ as $\sigma(T)$ is compact (so in particular closed). Hence, it is sufficient to prove that there is nothing more.

Let $\lambda \notin K \Rightarrow$ there is $\delta > 0$ s.t. $\inf_{k \in K} |\lambda - k| \geq \delta$. and we claim that T is invertible.

- $(T - \lambda I)u = \sum_{j \geq 1} (k_j - \lambda) \langle u, e_j \rangle e_j = 0 \Rightarrow \langle u, e_j \rangle = 0 \Rightarrow u = 0$ as $\{e_j\}$ are linearly independent $\Rightarrow$ injectivity

- $(T-\lambda I)$ is surjective: let $y \in H = L^2(0,1)$ and we have to find $u \in H = L^2(0,1)$ such that

$$\sum_{j\geq 1} (k_j - \lambda) \langle u, e_j \rangle e_j = \sum_{j \geq 1} \langle y, e_j \rangle e_j$$

We take $u = \sum_{j\geq 1} \frac{\langle y, e_j \rangle}{(k_j - \lambda)} e_j$. Clearly, $u \in H$ as

$$\sum_{j\geq 1} \left(\frac{\langle y, e_j \rangle}{k_j - \lambda} \right)^2 \leq \sum_{j \geq 1} \frac{\langle y, e_j \rangle^2}{(k_j-\lambda)^2} \overset{\delta \leq (k_j - \lambda)}{\leq} \frac{1}{\delta^2} \sum_{j\geq 1} \langle y, e_j \rangle^2 \leq$$

$$\leq \frac{\|y\|^2}{\delta^2} \text{ by Bessel's inequality.}$$

Hence $(T-\lambda I)$ is bounded isomorphism $\Rightarrow (T-\lambda I)^{-1}$ is bounded. II.

S15

This is useful rule in case we don't need to find point, continuous and residual parts of the spectrum (i.e. we are not interested WHY operator is not invertible).

Claim: $\exists_{h_n \in H} \exists_{\varepsilon_n \to 0}$ $\|A h_n\| \leq \varepsilon_n \|h_n\| \Rightarrow A$ does not have bdd inverse.

Proof: Suppose it has. Let $g_n = A h_n$, $h_n = A^{-1} g_n$. Then

$$\frac{1}{\varepsilon_n} \leq \frac{\|A^{-1} g_n\|}{\|g_n\|} \Rightarrow \text{contradiction as } \frac{1}{\varepsilon_n} \to \infty.$$

Note: In particular, A does not have inverse, otherwise we would get contradiction with Inverse Mapping Theorem.

(S12) Let $A = \overline{\{g(x) : x \in \mathbb{R}\}}$. If $\lambda \notin A$, there is $\delta > 0$ s.t.

$|\lambda - g(x)| \geq \delta \ \forall_{x \in \mathbb{R}}$. Then:

- G is injective: $(g(x)-\lambda) f(x) = 0 \Rightarrow f = 0$ a.e. $x \in \mathbb{R}$ as $g(x) - \lambda$ never vanishes
- G is surjective: the inverse is $Hf = \frac{1}{g(x)-\lambda} f(x)$. $H: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as $\int_{\mathbb{R}} \left| \frac{f(x)}{g(x)-\lambda} \right|^2 \leq \frac{1}{\delta^2} \int_{\mathbb{R}} |f(x)|^2$.

$\Rightarrow$ G has inverse and since G is bdd ~~on~~ $L^2(\mathbb{R})$, the inverse is bdd btw. Hence $A \supset \sigma(G)$.

Now, let $\lambda \in A$. We prove $\lambda \in \sigma(G)$. First, we assume $\exists_{y \in \mathbb{R}}$ s.t. $\lambda = g(y)$. As g is continuous, $\forall_{\varepsilon > 0} \exists_{\delta > 0}$ $|x-y| \leq \delta_\varepsilon \Rightarrow |g(x) - g(y)| \leq \varepsilon$.

Let $f_\varepsilon = \mathbb{1}_{|x-y| \leq \delta_\varepsilon}$. Then $(G - \lambda) f_\varepsilon(x) = \underbrace{(g(x) - g(y))}_{\leq \varepsilon \text{ as we are on } |x-y| \leq \delta_\varepsilon} f_\varepsilon(x)$

$\Rightarrow \|(G-\lambda I) f_\varepsilon\|_{L^2} \leq \varepsilon \|f_\varepsilon\|_{L^2}$. By (S15), $G - \lambda I$ cannot be invertible. Hence, $\text{int}\, A \subset \sigma(G)$.

Finally, $\sigma(G)$ is compact, in particular it is closed so $A = \overline{\text{int} A} \subset \sigma(G)$.

Conclusion: $\sigma(G) = A$, as desired.

(513)

We study right shift operator on $\ell^2(\mathbb{Z})$. $(Rx)_k = x_{k-1}$.

(a) point spectrum: let $\lambda \in \mathbb{C}$

$$(R-\lambda I)x = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) - \lambda(\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \dots) =$$

$$= (\dots, x_{-1}-\lambda x_{-2}, x_0 - \lambda x_{-1}, x_1 - \lambda x_0, \dots)$$

$$= (x_{k+1} - \lambda x_k) = 0 \quad \Rightarrow \quad x_{k+1} = \lambda x_k$$

If $\lambda = 0 \Rightarrow x = 0$ and we are done. Otherwise, x is of the form $\left(\dots, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1, \lambda, \lambda^2, \dots\right) x_0$. But such x cannot be in $\ell^2(\mathbb{Z})$.

(b) If $|\lambda| \neq 1$, we can write inverse explicitly using series representations.

- $R - \lambda I = -\lambda\left(I - \frac{R}{\lambda}\right) \longrightarrow$ invertible for $|\lambda| > 1$ since $\|R\| = 1$ and $\left\|\frac{R}{\lambda}\right\| = \frac{1}{|\lambda|} < 1 \longrightarrow$ use series.

- $R - \lambda I = \underbrace{R}_{\text{invertible}}\underbrace{(I - \lambda R^{-1})}_{\text{invertible for } |\lambda|<1 \text{ as } \|R^{-1}\| = 1} =$ invertible for $|\lambda| < 1$.

left shift

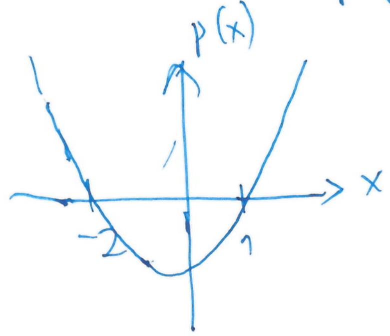
Here $R^{-1} = L =$ left shift.

So we have proved even more, $(R - \lambda I)$ is a bounded isomorphism for $|\lambda| \neq 1$.

(c) solution on page (17) (Hint: to prove density, use $\operatorname{im}(R-\lambda I) = \ker((R-\lambda I)^*)^\perp$.

S16

We already know that $\sigma(M) = [0,1]$. To find spectrum of M^2+M-2 we note that this is image of M under the polynomial $p(x) = x^2+x-2 = (x+2)(x-1)$



Since $p(0) = -2$, $p([0,1]) = [-2,0]$ (see picture).

Using S5 we conclude that

$$\sigma(M^2+M-2) = \sigma(p(M)) = p(\sigma(M)) = [-2,0].$$

$\square$.

Solution to S13 (c): From (a) we know that point spectrum is empty. We study existence of residual spectrum. From (b) we know that only $\{|\lambda|=1\} \supset \sigma(R)$. Let L be the left shift.

$$\overline{\operatorname{im}(R-\lambda I)} \underset{\substack{\uparrow \\ R^*=L}}{=} \ker(L-\lambda I)^\perp \underset{\uparrow}{=} \{0\}^\perp = H.$$

$\ker(L-\lambda I) = \{0\}$ as point spectrum of L is similarly empty as in the case of R

This shows that $\boxed{\sigma(R) = \sigma_c(R)}$

To see that $\sigma(R) = \{|\lambda|=1\}$, we apply S15. Fix $\lambda \in \{|\lambda|=1\}$.

Consider $x_n = (\ldots, 0, \bar\lambda, \bar\lambda^2, \bar\lambda^3, \ldots, \bar\lambda^n, 0, \ldots)$

$\lambda x_n = (\ldots, 0, 1, \bar\lambda, \bar\lambda^2, \ldots, \bar\lambda^{n-1}, 0, \ldots)$

$R x_n = (\ldots, 0, 0, \bar\lambda, \bar\lambda^2, \ldots, \bar\lambda^{n-1}, \bar\lambda^n, 0, \ldots)$

$$\Rightarrow (R-\lambda I)x_n = (\ldots, 0, -1, 0, 0, \ldots, 0, \bar{\lambda}^n, 0, \ldots)$$
$$x_n = (\quad \bar{\lambda}, \bar{\lambda}^2, \ldots, \bar{\lambda}^n, 0, 0, \ldots)$$

$$\Rightarrow \|(R-\lambda I)x_n\|_{l^2}^2 = 2, \quad \|x_n\|_{l^2}^2 = n$$

$$\Rightarrow \|(R-\lambda I)x_n\|_{l^2}^2 = \frac{2}{n}\|x_n\|_{l^2}^2 \Rightarrow \|(R-\lambda I)x_n\|_{l^2} \leq \sqrt{\frac{2}{n}}\,\|x_n\|_{l^2}$$

and $\sqrt{\frac{2}{n}} \to 0$.

(The idea of S15 is to find approximate spectrum: ~~[illegible]~~ here we could consider $y_n = \frac{x_n}{\sqrt{n}}$. Then $\|y_n\|_{l^2} = 1$, $(R-\lambda I)y_n \to 0$).

Implication in S9 ($\sigma_c \subset \sigma_{app}$) by M. Lasica

Let $\lambda \in \sigma_c(A)$, i.e. $\overline{\mathrm{im}(A-\lambda I)} = H$ but $\mathrm{im}(A-\lambda I) \neq H$. Take sequence x_n s.t. $(A-\lambda I)x_n \longrightarrow y \in H \setminus \mathrm{im}(A-\lambda I)$, x_n converges strongly.

Claim: $\|x_n\| \to \infty$. Proof: If not, $\|x_n\| \leq C$ for some constant C, independently of n. Choose weakly convergent subsequence $\{x_{n_k}\} \subset \{x_n\}$, $x_{n_k} \rightharpoonup x$. We observe that $Ax_{n_k} = \underbrace{(A-\lambda I)x_{n_k}}_{\to y} + \underbrace{(\lambda I)x_{n_k}}_{\rightharpoonup \lambda x}$ converges weakly to $y + \lambda x$.

It follows that $\underbrace{(A-\lambda I)x_{n_k}}_{\in \mathrm{im}(A-\lambda I)} \rightharpoonup y \in \mathrm{im}(A-\lambda I)$ as $\mathrm{im}(A-\lambda I)$ is convex.

Contradiction. $\square$.

Finally, we set $z_n = \frac{x_n}{\|x_n\|}$. Then $\|z_n\| = 1$, $\|(A-\lambda I)z_n\| \longrightarrow 0$ so $\lambda \in \sigma_{app}(A)$.

$\square$.

Problems from the lecture:

A1 It follows from the solution of R8 that we have always $\overline{M} = (M^\perp)^\perp$. Since $M^\perp = \{0\}$, we have that $(M^\perp)^\perp = H$ so that $\overline{M} = H$ as desired.

A2 Indeed, let $M = \{x \in H : \|x\| = 1\}$ be the unit sphere which is clearly not dense. We claim that $M^\perp = \{0\}$.

$\supseteq$: this is trivial

$\subseteq$: suppose there is $y \neq 0$, $y \in M^\perp$ i.e. $\forall_{m \in M} \langle y, m \rangle = 0$

Let $m = \frac{y}{\|y\|} \in M$ so that $0 = \langle y, \frac{y}{\|y\|} \rangle = \|y\|$

$\Rightarrow$ $y = 0$ contradiction.

$\square$