Functional Analysis (WS 19/20), Problem Set 9

(spectral theory of compact self–adjoint operators)

In what follows, let H be a **complex** Hilbert space and E, F be **complex** Banach spaces.

Let $T : H \to H$ be a bounded linear operator. We say that T is self-adjoint if $T = T^*$.
Let $T : E \to F$ be a linear operator. We say that T is compact if $\overline{T(B_1(0))}$ is compact in F.

Self-adjoint operators

S1. ♣ Prove that if $T : H \to H$ is self-adjoint then $\sigma(T)$ is real and ~~nonnegative~~[1].

S2. ♣ Prove that if $T : H \to H$ is self-adjoint then its eigenvectors corresponding to different eigenvalues are orthogonal.

S3. Let $M \subset H$ be a closed subspace. Recall what is the adjoint of the orthogonal projection on M denoted with P_M? What is $\sigma(P_M)$ and what are components of this spectrum (point, continuous, residual)?

S4. Let $M : L^2(0,1) \to L^2(0,1)$ be a multiplication operator defined with $Mf(x) = xf(x)$ cf. Problem S7 (PS 8). Prove that M is self-adjoint. Recall what is the spectrum of M.

S5. More generally, let g be a bounded, continuous function and consider multiplication operator $G : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined with $Gf(x) = g(x)f(x)$, cf. Problem S12 (PS 8). Recall what is the spectrum of G. Find sufficient and necessary condition on g so that G is a self-adjoint operator.

Compact operators

C1. ♣ Prove that if $T : X \to X$ is compact then T is bounded.

C2. ♣ Prove that the following are equivalent

(A) $A \subset X$ is bounded then $\overline{T(A)}$ is compact,

(B) $\overline{T(B(0,1))}$ is compact,

(C) if $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, there is a convergent subsequence in $\{Tx_n\}_{n\in\mathbb{N}}$.

C3. ♣ Let H be a ~~separable~~ Hilbert space. Prove that $T : H \to H$ is compact if and only if there is a sequence of finite rank operators $T_n : H \to H$ such that $T_n \to T$ in the operator norm.

C4. Let $g \in C[0,1]$ and $T : C[0,1] \to C[0,1]$ be defined with the formula $Tf(x) = \int_0^x f(t)g(t)dt$. Prove that T is a compact operator.

C5. For which Banach spaces X, the identity operator on X is compact?

C6. ♣ The purpose of this Problem is to study Hilbert-Schmidt operators (known in this lecture as *integral operators*) with kernels of lower regularity. More precisely, let $K \in L^2(\Omega \times \Omega)$ be a measurable kernel on $\Omega \times \Omega$. We define Hilbert-Schmidt operator $T : L^2(\Omega) \to L^2(\Omega)$ with

$$Tf(x) = \int_\Omega K(x,y)f(y)\,dy.$$

[1] This is a particularly stupid mistake. Even multiplication operator G from the Problem S5. with a sign-changing real-valued function g can serve as a counterexample

It was discussed in the lecture that if K is continuous then T is a compact operator. Apply Banach-Alaoglu-Bourbaki Theorem in the separable Hilbert space $L^2(\Omega)$ to deduce that if $K \in L^2(\Omega \times \Omega)$ then T is still a compact operator.

C7. ♣ Prove that if $T : H \to H$ is compact and H is infinite dimensional then $0 \in \sigma(T)$.

C8. ♣ The target of this exercise is to prove spectral characterization of compact operators. Let $T : H \to H$ be compact and let $\lambda \in \sigma(T)$ be such that $\lambda \neq 0$. Prove that

(a) image of $(I - T)$ is closed,

(b) every $\lambda \in \sigma(T)$ such that $\lambda \neq 0$ belongs to the point spectrum of T (*Hint:* Aiming at contradiction, consider subsets $(I - \lambda T)^n H$ and apply Riesz Lemma),

(c) there is m such that $\ker(T - \lambda I)^m = \ker(T - \lambda I)^{m+1}$ and this subspace is finite dimensional,

(d) the only accumulation point of $\sigma(T)$ can be 0,

(e) $\sigma(T)$ is at most countable.

A simple proof is given in the Wikipedia article: https://en.wikipedia.org/wiki/Spectral_theory_of_compact_operators.

C9. Find $\sigma(T)$ where $T : L^2(0,1) \to L^2(0,1)$ is given with the formula $Tf(x) = \int_0^x f(y)dy$.

C10. Find all bounded and continuous functions g such that the multiplication operator G from Problem S5 is compact.

C11. Consider the problem

$$x''(t) = f(t) \text{ for } t \in (0,1), \qquad x(1) = x'(0) = 0.$$

(a) Prove that for any $f \in L^2(0,1)$, there is a unique $x \in L^2(0,1)$ solving the problem above.

(b) We write $x = Kf$. Prove that $K : L^2(0,1) \to L^2(0,1)$ is a well-defined integral operator.

(c) Prove that K is a compact operator.

Spectral theorem for self-adjoint compact operators

Theorem: Let H be a separable Hilbert space and $A : H \to H$ be a compact self-adjoint operator. Then, there is a countable orthogonal basis of H (in the sense of Schauder) consisting of eigenvectors of A. Moreover, the corresponding eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ are real. If $\dim H = \infty$, $\lambda_k \to 0$ as $k \to \infty$.

Note that in view of Problems S1. and C8., only the spanning property is not clear.

ST1. **(roots of operators)** Let $A : H \to H$ be self-adjoint and compact linear operator on a separable Hilbert space H. Let $n \in \mathbb{N}$. Prove that there exists a a bounded linear operator $B : H \to H$ such that $B^n = A$.

ST2. **(approximate inverse)** Let $A : H \to H$ be a self-adjoint and compact linear operator on a separable Hilbert space H. Suppose that $\ker A = \{0\}$. Prove that there exists a sequence of operators $\{A_n\}_{n \in \mathbb{N}}$ such that $A_n A x \to x$ as $n \to \infty$.

ST3. Consider Radamacher system: $r_0 = 1$ and $r_n = \text{sgn}(\sin(2^n \pi t))$. Prove that this is an orthogonal system but it is not an orthogonal basis of $L^2(0,1)$.