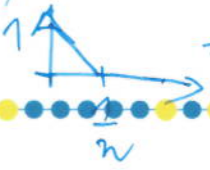
# AF: midterm review + more on Hilbert spaces

(H1) First, we check that $G \subset (G^\perp)^\perp$. Indeed, if $x \in G$, $\forall_{y \in G^\perp}$ we have $\langle y, x \rangle = 0$, hence $x$ is orthogonal to $G^\perp$ as desired.
Conversely, let $x \in (G^\perp)^\perp$. As $G$ is closed subspace, there is decomposition $x = u + v$ where $u \in G$, $v \in G^\perp$. In particular, $u \in (G^\perp)^\perp$.

$\Rightarrow \underbrace{x - u}_{\in (G^\perp)^\perp \text{ as } (G^\perp)^\perp \text{ is linear space}} = v \Rightarrow v \in (G^\perp)^\perp$. But $v \in G^\perp$. $\Rightarrow$ $\underline{v = 0}$

$\Rightarrow x = u \in G. \Rightarrow x \in G.$ ✓

(H2) Note that $e_i \in G$ hence $G^\perp = \{0\}$ and therefore $(G^\perp)^\perp = \ell^2$.
Previous result does not apply as $G$ is not closed (actually as $\{e_i\}_i$ is Schauder basis of $\ell^2$ (see Problem Set 1) $G$ is dense in $\ell^2$).

(H3) This is important application of Corollary of Inverse Mapping Theorem.
If you have two norms $\|\cdot\|_A$, $\|\cdot\|_B$ on $X$ making $X$ Banach space and $\|f\|_A \le C\|f\|_B$ then $\|f\|_B \le \tilde{C}\|f\|_A$. (see 03 in Problem Set 4).
Therefore, if $(C[0,1], \|\cdot\|_2)$ were Banach space, $\|f\|_\infty \le C\|f\|_2$ for some constant $C$. Take  $= f_n$. Then $\|f_n\|_\infty = 1$, $\|f_n\|_2 \to 0$.

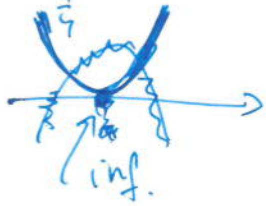
$\Rightarrow$ contradiction.

(1-14) $V = \{ f \in L^2(0,1) : \ f|_{[\frac{1}{4},\frac{3}{4}]} = \text{const} \}$. $\qquad A = [0,\frac{1}{4}] \cup [\frac{3}{4},1]$

$$\inf_{g\in V} \|f-g\|_2^2 = \inf_{g\in V} \left\{ \int_0^1 |f-g|^2 \right\} = \inf_{g\in V} \left\{ \int_A |f-g|^2 + \int_{\frac{1}{4}}^{\frac{3}{4}} |f-g|^2 \right\}$$

$$= \inf_{\substack{g\in V \\ g=f \text{ on } A}} \int_{\frac{1}{4}}^{\frac{3}{4}} |f-g|^2 \underset{\substack{\uparrow \\ g \text{ constant} \\ \text{on } [\frac{1}{4},\frac{3}{4}]}}{=} \inf_{a\in\mathbb{R}} \int_{\frac{1}{4}}^{\frac{3}{4}} |f(x)-a|^2 dx$$

Consider $F(a) = \int_{\frac{1}{4}}^{\frac{3}{4}} |f(x)-a|^2$. $\quad F'(a) = 2\int_{\frac{1}{4}}^{\frac{3}{4}} (f(x)-a) = 0$

$\uparrow$ this is polynomial 

$$\Rightarrow a = 2\int_{1/4}^{3/4} f(x)$$

$$\Rightarrow P_V f = \begin{cases} f & \text{on } A \\ 2\int_{1/4}^{3/4} f & \text{on } [\frac{1}{4},\frac{3}{4}] \end{cases} \in V.$$

$$\Rightarrow P_{V^\perp} f = \begin{cases} 0 & \text{on } A \\ f - 2\int_{1/4}^{3/4} f & \text{on } [\frac{1}{4},\frac{3}{4}] \end{cases} \in V^\perp.$$

Therefore, $V^\perp = \{ f \in L^2(0,1) : \ f|_A = 0, \ \int_{[\frac{1}{4},\frac{3}{4}]} f = 0 \}$.

(H5)
$$\sum_{n=1}^{\infty} \left| \int_0^t x^3 f_n(x)\,dx \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \mathbb{1}_{[0,t]} x^3 f_n(x)\,dx \right|^2 =$$

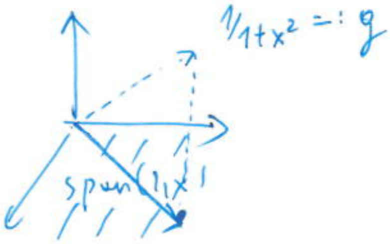
$$= \sum_{n=1}^{\infty} \left| \langle \mathbb{1}_{[0,t]} x^3, f_n \rangle \right|^2 \underset{\substack{\uparrow \\ \text{Parseval's} \\ \text{identity}}}{=} \| \mathbb{1}_{[0,t]} x^3 \|_{L^2}^2 = \int_0^t x^6\,dx$$

$$= \frac{1}{7} t^7. \checkmark$$

(H6)
$$X = \left\{ f \in L^2(-1,1) : \int_{-1}^1 f(x) = \int_{-1}^1 x f(x) = 0 \right\} =$$

$$= \left\{ f \in L^2(-1,1) : f \perp \mathrm{span}(1,x) \right\} = \{ \underbrace{\mathrm{span}(1,x)}_{\substack{\uparrow \text{ finite dimensional} \\ \text{hence, closed subspace}}} \}^{\perp}$$

In particular, $X^{\perp} = \mathrm{span}(1,x)$ by H1.
It may be hard to find projection on $X$ but it is easy to find projection on $X^{\perp}$.




$$g - P_{X^{\perp}} g \perp 1 \Rightarrow \int_{-1}^1 \frac{1}{1+x^2} - a - bx = 0$$
$$g - P_{X^{\perp}} g \perp x \Rightarrow \int_{-1}^1 \left( \frac{1}{1+x^2} - a - bx \right) x = 0$$

$$P_{X^{\perp}} g = a + bx$$

$a, b = ?$

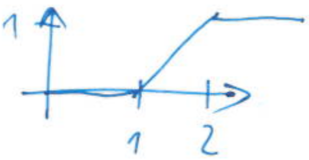
$$\Downarrow$$

$$\frac{\pi}{2} - 2a = 0 \Rightarrow a = \frac{\pi}{4}$$
$$0 - b = 0 \Rightarrow b = 0$$

$$\Rightarrow P_{X^{\perp}} g = \frac{\pi}{4} \Rightarrow P_X g = \frac{1}{1+x^2} - \frac{\pi}{4} = g - \frac{\pi}{4}$$

$$\mathrm{dist}(g, X) = \| g - P_X g \|_{L^2} = \left\| \frac{\pi}{4} \right\|_{L^2(-1,1)} = \frac{\pi}{4} \left( \int_{-1}^1 1 \right)^{1/2} = \frac{\sqrt{2}\,\pi}{4}. \checkmark$$

R1 Clearly, this is a normed space. We claim it is not Banach.

Let $f(t)=$  $\notin C^1$

Consider $F_n(s) = \frac{1}{n}\int_s^{s+\frac{1}{n}} f(t)\,dt$. By continuity of $f$, $F_n(s) \longrightarrow f(s)$ uniformly on $\mathbb{R}$ (but it is not necessary: it is sufficient that $F_n(0) \to f(0)$). Also $F_n'(s) = \frac{f(s+\frac{1}{n}) - f(s)}{n} \longrightarrow \begin{cases} 0 & s<1 \\ 1 & s\in(1,2) \\ 0 & s>2 \end{cases}$

Hence, $F_n'(s) \longrightarrow \mathbb{1}_{(1,2)}$ a.e. and by Dominated Convergence Theorem

$$\| F_n'(s) - \mathbb{1}_{(1,2)} \|_{L^1} \longrightarrow 0 \Rightarrow (F_n(0), F_n'(t)) \text{ is a Cauchy}$$

sequence in $\mathbb{R}\times L^1(\mathbb{R}) \Rightarrow F_n$ is Cauchy sequence in $X$

But $F_n$ does not have limit in $X$ as $F \notin C^1$.

R2 $(Tx)_n = \sum_{k=n}^{\infty} x_k$ Note that $T: \ell^1 \to c_0$ is well-defined

as $\sum_{k=1}^{\infty} |x_k| < \infty$ implies $\lim_{n\to\infty} \sum_{k=n}^{\infty} x_k = 0$ so $Tx \in c_0$.

Clearly $\underset{\substack{\uparrow \\ \text{norm in } c_0 \\ \text{(it is always considered} \\ \text{as subspace of } \ell^\infty)}}{\|(Tx)\|_\infty} = \sup_n |(Tx)_n| = \sup_n \left| \sum_{k=n}^{\infty} x_k \right| \le$

$$\le \sup_n \left\{ \sum_{k=n}^{\infty} |x_k| \right\} \le \sum_{k=1}^{\infty} |x_k| = \|x\|_1 \Rightarrow \|T\| \le 1.$$

But $Te_1 = (1,0,0,\dots)$, $\|Te_1\|_\infty = 1$, $\|e_1\|_{\ell^1} = 1$

so $\|T\| = 1$ (norm is even attained)

R3 $T_n(g) := \int_0^1 f_n(t)\, g(t)\, dt$. $T_n \in (L^2(0,1))^*$ by Hölder inequality

$\forall_{g \in L^2} \sup_n |T_n(g)| \leq C(g)$ as we know that for each $g \in L^2$

$$\lim_{n \to \infty} T_n(g) = 0.$$

By Banach-Steinhaus $\sup_{\|g\|_2 \leq 1} \sup_n \left| \int_0^1 f_n(t)\, g(t)\, dt \right| \leq C$

for some constant $C$. Take $g = f_n / \|f_n\|_2$ so we get

$$\sup_n \left| \int_0^1 f_n(t) \frac{f_n(t)}{\|f_n\|_{L^2}}\, dt \right| \leq C \Rightarrow \sup_n \|f_n\|_{L^2} \leq C$$

as desired.

<u>But</u> it is not true that $\lim_{n \to \infty} \|f_n\|_2 = 0$. Let $\{f_n\}$ be orthonormal basis of $L^2(0,1)$. As series $\sum_n (f_n, g)^2$ is convergent for all $g \in L^2$ by Bessel's inequality $(f_n, g) \longrightarrow 0$ as $n \to \infty$ (necessary condition of series convergence).

On the other hand, $\|f_n\|_{L^2} = 1 \quad \forall_{n \in \mathbb{N}}$.

R4 Let $T: \ell^1 \to \ell^1$ be given with $(Tx)_i = \sum_{j=i}^{\infty} \frac{x_j}{2^{j+i}}$

Then $\|Tx\|_{\ell^1} = \sum_{i=1}^{\infty} |(Tx)_i| \leq \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \frac{x_j}{2^{j+i}} \leq$

$$\leq \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \frac{|x_j|}{2^{2i}} \leq \|x\|_{\ell^1} \sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \|x\|_{\ell^1} \underbrace{\sum_{i=1}^{\infty} \frac{1}{4^i}}_{\leq \frac{1}{2}}$$

$$\leq \frac{1}{2} \|x\|_{\ell^1} \Rightarrow \quad \|T\| \leq \frac{1}{2} < 1$$

By standard theory, $I-T$ is invertible (here we use $\|T\|<1$)

Now, our equation can be written as $x = Tx + y$ which is the same as $x - Tx = y \iff (1-T)x = y$

As $(1-T)$ is bounded operator and bijection this is equivalent to $x = (1-T)^{-1} y$. The conclusion follows.