Functional Analysis (WS 20/21)

(Special Problems)

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<u>Rules:</u> Each problem has assigned deadline for submission of the solution. If the problem remains unsolved, the deadline is extended and some hints are provided. Each problem is worth 4 points in the tutorial classification (added independently of regular homeworks, active class participation).

Contents

1 Invitation to Sobolev spaces.

Announced: 16/10/2020, Deadline: 13/11/2020.

One of the most fundamental topic in analysis is the notion of Sobolev spaces and weak derivatives. Let $f \in L^p(0,1)$ where $1 \le p \le \infty$. We say that f is weakly differentiable if there is a function $g \in L^p(0,1)$ such that

$$\int_0^1 f(x)\,\phi'(x)\,\mathrm{d}x = -\int_0^1 g(x)\,\phi(x)\,\mathrm{d}x$$

for all $\phi \in C_c^\infty(0,1)$ (i.e. smooth functions $\phi : [0,1] \to \mathbb{R}$ with a compact support in $(0,1)$). If this is the case, we write $f' = g$ and we say that g is the weak (Sobolev) derivative of f. The space of all weakly differentiable functions in this sense is denoted with $W^{1,p}(0,1)$ and is called Sobolev space.

In what follows, we gonna check that weak derivatives make sense (they coincide with strong derivatives whenever the latter exist) and they don't see what happens on small sets (i.e. sets of measure zero). Finally, functional analytic properties of Sobolev spaces will be established.

(A) Prove that weak derivatives are uniquely defined, up to a set of measure zero.

(B) Suppose that $f \in C^1[0,1]$. Prove that classical and weak derivatives of f coincide.

(C) Give an example of function $f \in L^p(0,1)$ such that $f \notin C^1(0,1)$ but $f \in W^{1,p}(0,1)$.

(D) Prove that if $f \in W^{1,p}(0,1)$ and $f' = 0$ a.e. in $(0,1)$ then f is constant a.e., i.e. there is $C \in \mathbb{R}$ such that $f(x) = C$ for a.e. $x \in (0,1)$. (Hint: convolution with approximate identity.)

(E) Prove that $W^{1,p}(0,1)$ is a Banach space equipped with the norm

$$\|f\|_{1,p} := \|f\|_p + \|f'\|_p.$$

Sobolev spaces allow to differentiate functions which are slightly less regular than C^1. All results for classical derivatives (chain rule, product rule, integration by parts) have their analogue for weak derivatives. But it is somehow easier to be in $W^{1,p}(0,1)$ rather than in $C[0,1]$ and this is the reason people in analysis and PDEs prefer to work with Sobolev spaces.

2 Real interpolation in L^p spaces.

Announced: 16/10/2020, Deadline: 27/11/2020.

The following (simplified) *interpolation* result is crucial in harmonic analysis. Briefly speaking it allows to deduce that when $T : L^p \to L^p$ and $T : L^q \to L^q$ then $T : L^r \to L^r$ for $r \in (p,q)$ (so we interpolate between p and q). The result is a major step in the proof of boundedness of Hardy-Littlewood maximal function (which implies Lebesgue Differentiation Theorem) and study of Hilbert transform.

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. We (as always) write L^p for $L^p(X, \mathcal{F}, \mu)$. Suppose that T is a (nonlinear) operator defined on measurable functions into measurable functions. Assume that T is sublinear i.e. there is a constant $c > 0$ such that

$$|T(f+g)| \le c\,|Tf| + c\,|Tg|, \qquad |T(\lambda f)| = |\lambda|\,|Tf| \text{ for all } \lambda \in \mathbb{R}.$$

Moreover, assume that T is a bounded operator as a map from L^p to L^p and from L^q to L^q, i.e. there are constants C_p, C_q such that

$$\|Tf\|_p \le C_p\,\|f\|_p, \qquad \|Tf\|_q \le C_q\,\|f\|_q.$$

Prove that T is a bounded operator from L^r to L^r for all $r \in (p,q)$, i.e. for all $r \in (p,q)$, there is a constant C_r such that

$$\|Tf\|_r \le C_r\,\|f\|_r.$$

Here are some loosely written hints:

(A) If $f \in L^p$ and $1 \le p_0 < p < p_1 \le \infty$ one can find $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ such that $f = f_0 + f_1$.

(B) If $\varphi : [0,\infty) \to \mathbb{R}$ is an increasing and differentiable function with $\varphi(0) = 0$ then

$$\int_X \varphi(|f|(x))\,d\mu(x) = \int_0^\infty \varphi'(t)\,\mu(|f| > t)\,dt.$$

In particular, consider $\varphi(t) = t^p$.

3 Laplace (Poisson) equation by a simple version of Lax-Milgram Lemma.

Announced: 1/12/2020, Deadline: 28/01/2021.

In the following we apply Lax-Milgram Lemma (known from the tutorials) to study the most famous PDE

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^n \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$ is given, $\Delta = \partial^2_{x_1} + ... + \partial^2_{x_n}$, Ω is a smooth bounded domain (say, a ball) and $u : \Omega \to \mathbb{R}$ is the desired function.

In the modern language, this equation is understood as follows. Let $H^1_0(\Omega) := \overline{C^\infty_c(\Omega)}$ where the closure is taken with respect to the $W^{1,2}(\Omega)$ norm. This space represents functions in $W^{1,2}(\Omega)$ that vanish at the boundary $\partial\Omega$. Then, we say that $u \in H^1_0(\Omega)$ is a weak solution to (1) provided that for all $\varphi \in C^\infty_c(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\, \varphi.$$

Follow the steps below to prove that there exists the unique weak solution to (1) and this weak solution concept makes sense.

(A) Prove that if u is a strong solution to (1) ($u \in C^2(\overline{\Omega})$ and it satisfies (1) pointwisely) then u is also a weak solution.

(B) Prove Poincare inequality: there is a constant C such that for all $u \in H^1_0(\Omega)$ we have

$$\|u\|_2 \leq C\, \|\nabla u\|_2.$$

Deduce that $\|\nabla u\|_2$ defines an equivalent norm on $H^1_0(\Omega)$. (See FAQ below for some simplifications.)

(C) Prove that weak solutions can be equivalently defined for test functions $\varphi \in H^1_0(\Omega)$.

(D) Define appropriate symmetric bilinear form on $H^1_0(\Omega)$ and apply symmetric Lax-Milgram Lemma.

Motivation for weak solutions: Of course, one would like to solve (1) in the strong sense, i.e. to find C^2 function u such that (1) holds. However, this is in general very hard. Therefore, the process is divided for two steps:

- find weak solutions (this is easier than finding directly strong solutions),
- upgrade weak solution to the strong one (this is also easier because we have something already in hand).

The second part is called *regularity theory* as we want to find regularity of u (say, at least C^2). In last years, regularity theory became a new field on its own. For instance, a deep result, due to Nash, de Giorgi and Moser, asserts that if $f \in L^\infty$ (so this is very weak assumption!!!), the unique weak solution is in $C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$. This solves XIXth Hilbert's Problem.

FAQ to Problem 3.

1. The partial weak derivative of $u \in W^{1,2}(\Omega)$ is defined as a function $u_{x_i} \in L^2(\Omega)$ such that

$$\int_\Omega u\,\varphi_{x_i} = -\int_\Omega u_{x_i}\,\varphi \text{ for all } \varphi \in C_c^\infty(\Omega).$$

2. Scalar dot in $\int_\Omega \nabla u \cdot \nabla \varphi$ means scalar product of two vectors:

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \Big[u_{x_1}\,v_{x_1} + u_{x_2}\,v_{x_2} + ... + u_{x_n}\,v_{x_n}\Big].$$

   Then, weak gradient of u is simply $\nabla u = (u_{x_1}, ..., u_{x_n})$.

3. If you struggle to prove Poincare inequality in the whole generality, prove it only in 1 dimension and $\Omega = (0,1)$. You can also attack the case of arbitrary dimension and $\Omega = (0,1)^d$ which is still easier than the case with arbitrary Ω.

4. What does $\|\nabla u\|_2$ in Poincare inequality mean? The meaning of $\|u\|_2$ is clear: this is simply L^2 norm of u. For $\|\nabla u\|_2$, we need to generalize L^2 norms for vector-valued functions. One possibility (which is convenient in this problem) is to define

$$\|\nabla u\|_2 := \left[\int_\Omega u_{x_1}^2 + ... + u_{x_n}^2\right]^{1/2}.$$

   Then, norm on $W^{1,2}(\Omega)$ can be defined as

$$\|u\|_{H^1} = \left[\|u\|_2^2 + \|\nabla u\|_2\right]^{1/2} = \left[\int_\Omega u^2 + u_{x_1}^2 + ... + u_{x_n}^2\right]^{1/2}.$$

5. To prove that weak solutions can be equivalently defined for test functions $\varphi \in H_0^1(\Omega)$ means you have to prove: u is a weak solution if and only if for all $\varphi \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\,\varphi.$$

4 Stationary Fokker-Planck equation and general Lax-Milgram Lemma.

Announced: 1/12/2020, Deadline: 28/01/2021.

Now, the plan is to generalize Lax-Milgram Lemma to study equations like

$$\begin{aligned} -\Delta u + \sum_{i=1}^{n} b_i\, u_{x_i} + \gamma\, u = f \text{ in } \Omega \subset \mathbb{R}^n \\ u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

where $f \in L^2(\Omega)$ and $b_i \in L^\infty(\Omega)$ are given, $\Delta = \partial^2_{x_1} + ... + \partial^2_{x_n}$, Ω is a smooth bounded domain (say, a ball), $\gamma \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ is the desired function. Again, we say that $u \in H^1_0(\Omega)$ is a weak solution to (2) provided that for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega b_i\, u_{x_i}\, \varphi + \gamma \int_\Omega u\, \varphi = \int_\Omega f\, \varphi.$$

(A) (**Lax-Milgram**) Let $(H, \|\cdot\|)$ be a Hilbert space. Suppose that $a : H \times H \to \mathbb{R}$ is a bilinear continuous form that is coercive, i.e. there is a constant $C > 0$ such that $a(u,u) \geq C\|u\|^2_H$. Moreover, let $l \in H^*$. Prove that there exists uniquely determined $u \in H$ such that

$$a(u,v) = l(v)$$

holds for all $v \in H$. *Hint:* Define appropriate map $A : H \to H^*$ and try to prove that A has closed image. Using orthogonal complements prove that A is surjective.

(B) Prove that there exists γ_0 such that for all $\gamma \geq \gamma_0$, (2) has the unique weak solution.

In estimates, it may be helpful to apply ε-Cauchy-Schwartz inequality:

$$|a\, b| \leq \varepsilon\, a^2 + \frac{1}{4\,\varepsilon}\, b^2.$$

5 Nonlinear equations and Stampacchia's Theorem.

Announced: 1/12/2020, Deadline: 28/01/2021.

The last generalization concerns some weakly nonlinear equations. The particular example we have in mind is

$$\begin{aligned} -\Delta u + g(u) &= f \text{ in } \Omega \subset \mathbb{R}^n \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous and increasing. I follow the formulation from Problem Set 2 in NPDE I course at UniBonn.

To obtain theory for such equations, we generalize Lax-Milgram Lemma to get:

Stampacchia's Theorem. Let H be a Hilbert space. Let $a : H \times H \to \mathbb{R}$. Assume that a satisfies

(1) for each $u \in H$, the map $v \mapsto a(u, v)$ is continuous and linear (it belongs to H^*),

(2) $|a(u_1, v) - a(u_2, v)| \leq \beta \|u_1 - u_2\| \|v\|$,

(3) $a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \geq \gamma \|u_1 - u_2\|^2$

for some constants β and γ. Then for every $l \in H^*$, there exists uniquely determined u such that $a(u, v) = l(v)$ for all $v \in H$.

We proceed as follows:

(A) Prove that if a (nonlinear!) map $A : H \to H$ satisfies

  (1) $\|A(u_1) - A(u_2)\| \leq \beta \|u_1 - u_2\|$,
  (2) $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \gamma \|u_1 - u_2\|^2$,

  then for every $f \in H$ there is a unique $u_f \in H$ such that $A(u_f) = f$. *Hint:* apply Banach Fixed Point Theorem to the map $R(u) = u - \lambda A(u) + \lambda f$ for appropriate λ.

(B) Prove Stampacchia's Theorem.

(C) Define weak solutions to (3). Prove that there exists the unique weak solution to (3).