AF Tutorial 13

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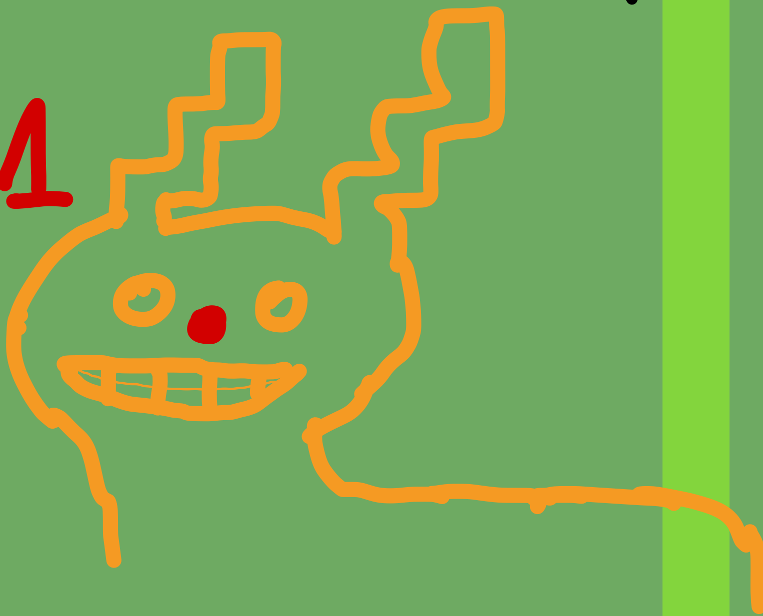

TRANSFORMATA FOURIERA

PRZESTRZENIE SCHWARTZA

$\mathcal{S}(\mathbb{R}^d)$: funkcje niesk. wiele razy różniczkowalne w sposób ciągły +

$$\sup_{x \in \mathbb{R}^d} x^{\alpha} \left| D^{\beta} f(x) \right| < \infty \qquad \forall_{\alpha, \beta}.$$

$$\sup_{x\in\mathbb{R}^d} |x^\alpha| \, |D^\beta f(x)| < \infty \qquad \forall \; \alpha\in\mathbb{N},\ \beta\in\mathbb{N}^d$$

$$\beta = (\beta_1, \dots, \beta_n)$$

$$D^\beta = \left(\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}\right)$$

Każda pochodna $f\in S(\mathbb{R}^d)$ znika szybciej przy $x\to\infty$ niż dowolny wielomian.

Przykłady:

(1) $e^{-|x|} \in S(\mathbb{R})$, $e^{-|x|^2} \in S(\mathbb{R})$

$\left(x^k e^{-|x|} \to 0 \quad x \to \infty \right)$

(2) $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$

S3 $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \underline{f \in L^p(\mathbb{R}^d)} \quad \forall_{1 \leq p \leq \infty}$

$\underline{D^\beta f \in L^p(\mathbb{R}^d)}$

$\underline{p = \infty}$ $\quad f \in L^\infty(\mathbb{R}^d)$ bo $\alpha = 0 = \beta$ $\quad |x|^\alpha \cdot |f| \leq C_\alpha$

$\underline{1 \leq p < \infty}$

$$\int_{\mathbb{R}^d} |f|^p \, dx = \int_{B(0,1)} |f|^p \, dx + \underbrace{\int_{\mathbb{R}^d \setminus B(0,1)} |f|^p \, dx}$$

$$\leq \|f\|_\infty^p \cdot |B(0,1)| + \int_{\mathbb{R}^d \setminus B(0,1)} \frac{C_\alpha^p}{|x|^{\alpha p}} \, dx$$

$$\leq \underbrace{\|f\|_\infty^p \cdot |B(0,1)|}_{<\infty} + \int_{\mathbb{R}^d \setminus B(0,1)} \frac{C_\alpha^p}{|x|^{\alpha p}}\, dx$$

$$\int_1^\infty \frac{1}{r^\alpha} < \infty \iff \alpha > 1.$$

$$= \text{———} \| \text{———} + C_\alpha^p \int_1^\infty \frac{1}{r^{\alpha p}} \cdot C_n \cdot r^{n-1}\, dr$$

$$= \text{———} \| \text{———} + C_\alpha^p \int_1^\infty 1 \cdot C_n \, r^{n-1-\alpha p}\, dr$$

$$n-1-\alpha p < -1 \iff n < \alpha p \iff \alpha > \frac{n}{p}.$$

$$\sup_{x\in\mathbb{R}^d} |x|^\alpha \, |D^\beta f| < \infty \qquad \forall_{\alpha,\beta}$$

$$\Rightarrow \exists \, C_{\alpha,\beta} \qquad \sup_{x\in\mathbb{R}^d} |x|^\alpha \, |D^\beta f| \le C_{\alpha,\beta}$$

$$C_\alpha = C_{\alpha,\emptyset}$$

DEF: Dla $f \in L^1(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \overbrace{f(x)}\, e^{-2\pi i \, \underbrace{\overset{\in \mathbb{R}^n}{\xi} \cdot \overset{\in \mathbb{R}^n}{x}}_{\text{iloczyn skalarny}}} \, dx$$

$$\hat{f} : \mathbb{R}^n \to \mathbb{C}.$$

transformata Fouriera $f : \mathbb{R}^n \to \underline{\mathbb{C}}.$

Zad. 1 $f \in L^1$

Riemann–Lebesgue Lemma.

(A) $\|\hat{f}\|_\infty \leq \|f\|_1$

(B) $\hat{f}$ jest ciągła

(C) $\hat{f}(\xi) \to 0$ gdy $|\xi| \to \infty$

D-d: $$\left| \int_{\mathbb{R}^n} f(x) \underbrace{e^{-2\pi i \xi \cdot x}}_{|\dots| \leq 1} dx \right| \leq \int_{\mathbb{R}^n} |f(x)| \, dx = \|f\|_1 .$$

to leży na okręgu jedn. w $\mathbb{C}$.

(B) $\hat{f}$ jest ciągła jako funkcja ξ.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx$$

$$\xi_n \to \xi \quad \overset{?}{\Longrightarrow} \quad \hat{f}(\xi_n) \to \hat{f}(\xi),$$

$$\Downarrow \qquad\qquad\qquad\qquad \Uparrow ?$$

$$e^{-2\pi i \xi_n \cdot x} \to e^{-2\pi i \xi \cdot x} \Longrightarrow \forall_x \; \underbrace{\left| f(x)\, e^{-2\pi i \xi_n \cdot x} \right|}_{\leq |f| \in L^1} \to f(x) e^{-2\pi i \xi x}$$

bo $e^{-\cdots}$ jest ciągłe

(C) gdy $|\xi| \to \infty$ to $|\hat{f}(\xi)| \to 0$.

$\underbrace{f \in C_c^\infty(\mathbb{R}^n)}$ najpierw taki przypadek.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \underbrace{e^{-2\pi i \xi x}}_{\partial_{x_i} e^{-2\pi i \xi x} \cdot \left(-\frac{1}{2\pi i \xi_i}\right)} dx =$$

$$= -\frac{1}{2\pi i \xi_i} \int_{\mathbb{R}^n} f(x)\, \partial_{x_i}\left(e^{-2\pi i \xi x}\right) dx = \quad \text{bo } \overset{C^\infty}{f} \in C_c^\infty$$

$$= +\frac{1}{2\pi i \xi_i} \int_{\mathbb{R}^n} \partial_{x_i} f \cdot \overbrace{e^{-2\pi i \xi x}}^{|\cdot| \leq 1} dx \leq \frac{1}{2\pi i \xi} \cdot \|\partial_{x_i} f\|_1$$

$\| \partial_{x_i} f \|_1 < \infty$ bo f ma zwarty nośnik np. K

$$\int_{\mathbb{R}^n} |\partial_{x_i} f| \leq \int_K |\partial_{x_i} f| \, dx \leq \|\partial_{x_i} f\|_\infty \cdot |K|, < \infty.$$

$f \in C_c^\infty(\mathbb{R}^n)$ to $|\hat{f}(\xi)| \leq \frac{1}{|2\pi i \xi|} C \to 0$

$\xi \to \infty$

Mamy że $\forall_{f \in C_c^\infty(\mathbb{R}^n)} \quad |\hat{f}(\xi)| \longrightarrow 0$ gdy $\xi \to \infty$.

$f \in L^1(\mathbb{R}^n)$

$C_c^\infty(\mathbb{R}^n)$ jest gęste w $L^1(\mathbb{R}^n) \Rightarrow \exists_{f_n} \; f_n \longrightarrow f$ w $L^1(\mathbb{R}^n)$

$$\underbrace{|\hat{f}(\xi)|}_{\to 0 \text{ przy } \xi \to \infty} \leq |\hat{f}(\xi) - \hat{f}_n(\xi)| + |\hat{f}_n(\xi)| \leq$$

$$\leq |(\hat{f} - \hat{f}_n)(\xi)| + |\hat{f}_n(\xi)|$$

$$\leq \|f - f_n\|_1 + |\hat{f}_n(\xi)|$$

$$|\hat{f}(\xi)| \leq \|f_n - f\|_1 + \underbrace{|\hat{f_n}(\xi)|}_{\to 0 \text{ gdy } \xi \to \infty}$$

$$\underbrace{\limsup_{\xi \to \infty} |\hat{f}(\xi)|}_{\text{lewa strona nie zależy od } n} \leq \underbrace{\|f_n - f\|_1}_{n \to \infty \text{ zbiega do } 0.}$$

$$\limsup_{\xi \to \infty} |\hat{f}(\xi)| \leq 0 \Rightarrow \hat{f}(\xi) \to 0$$

gdy $\xi \to \infty$.

② Załóż, że $f, g \in S(\mathbb{R}^d)$

(A) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$

<u>D-d</u>: $\displaystyle\int_{\mathbb{R}^n} f * g(x)\, e^{-2\pi i \xi \cdot x}\, dx =$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy\; e^{-2\pi i \xi \overset{(x-y)+(y)}{x}}\, dx$$

$$= \int\int f(y) g(x-y)\, e^{-2\pi i \xi (x-y)}\, e^{-2\pi i \xi y}\, dx\, dy$$

$$= \int\int \underbrace{f(y)}\,\underbrace{g(x-y)}\,\underbrace{e^{-2\pi i\xi(x-y)}}\,\underbrace{e^{-2\pi i\xi y}}\,dx\,dy$$

$$= \int f(y)\left[\int g(x-y)\,e^{-2\pi i\xi(x-y)}\,dx\right] e^{-2\pi i\xi y}\,dy$$

$= \hat{g}$ (przesunięcie o y)

$$= \hat{g}(\xi)\int f(y)\,e^{-2\pi i\xi y}\,dy = \underline{\hat{g}(\xi)\,\hat{f}(\xi)}.$$

(tw. Fubiniego $\displaystyle\int\int |f(y)|\,|g(x-y)|\,dx\,dy \leq \|f\|_2\,\|g\|_1 < \infty$

$= \||f| * |g|\|_1$ ← nier. Younga.

(B) $(T_h f)(x) = f(x+h)$ T_h = operator przesunięcia

$$\widehat{T_h f}(\xi) = \hat{f}(\xi)\, e^{2\pi i \xi \cdot h}$$

przesunięcie

$\Downarrow$

obrót

$$\widehat{T_h f}(\xi) = \int f(x+h)\, e^{-2\pi i \xi \cdot x}\, dx$$

$$= \int f(x+h)\, e^{-2\pi i \xi (x+h)}\, dx \quad e^{2\pi i \xi h}$$

$$= \hat{f}(\xi)\, e^{2\pi i \xi h}.$$

(c) $\widehat{f_{x_j}}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$

RÓŻNICZKOWANIE

$\Downarrow$

MNOŻENIE

PRZEZ WIELOMIAN

$$\widehat{\partial_{x_j} f}(\xi) =$$

$$= \int \partial_{x_j} f(x)\, e^{-2\pi i \xi x}\, dx =$$

$$= -\int f(x)\,(-2\pi i \xi_j)\, e^{-2\pi i \xi x}\, dx =$$

$$= 2\pi i \xi_j \hat{f}(\xi).$$

$$\int_0^1 f'(t)\, g(t) = -\int f(t)\, g'(t)\, dt$$

$$+ \quad f(t)\, g(t) \Big|_0^1$$

wyrazy brzegowe.

tu wyrazy brzegowe znikają bo

$$\partial_x f \in S(\mathbb{R}^d)$$

$$\widehat{\partial_{x_j} f}(\xi) = (2\pi i \xi_j)\,\hat{f}(\xi)$$

$$D^{\beta} f = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$$

$$\beta = (\beta_1, \dots, \beta_n)$$

$$\boxed{\widehat{D^{\beta} f}(\xi) = \overbrace{(2\pi i)^{\beta_1+\dots+\beta_n}\, \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}}\, \hat{f}(\xi).}$$

symbol Fourie
operatora różniczkowego!

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$$f(x) = e^{-\pi x^2} \Longrightarrow \hat{f}(\xi) = e^{-\pi x^2}.$$

$$f(x) = e^{-x} \mathbb{1}_{x \geq 0}$$

$$\int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-x} \mathbb{1}_{x \geq 0} e^{-2\pi i \xi x} dx =$$

$$= \int_0^\infty e^{-x} e^{-2\pi i \xi x} dx = \int_0^\infty e^{-(1+2\pi i \xi) x} dx$$

$$= \frac{1}{1+2\pi i \xi}$$

$$f(x) = e^{-x}\,\mathbb{1}_{x \geq 0} \in L^1(\mathbb{R})$$

$$\hat{f}(\xi) = \frac{1}{1 + 2\pi i \xi} \notin L^1(\mathbb{R})$$

__Tw wykł__: transformata Fouriera jest bijekcją na $S(\mathbb{R}^d)$ i operator odwrotny

$$\check{f}(x) = \int_{\mathbb{R}^d} f(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

$f \in S(\mathbb{R}^d) \Rightarrow \hat{f} \in S(\mathbb{R}^d)$

$f \in S(\mathbb{R}^d) \Rightarrow \check{f} \in S(\mathbb{R}^d).$

$\Rightarrow$ dystrybucje temperowane.

4) $f \in S(\mathbb{R}^n)$

Znaleźć wszystkie $u \in S(\mathbb{R}^n)$

$$-\overbrace{\underbrace{\Delta}_{=\sum_{i=1}^{n} \partial^2_{x_i}} u}^{\in S(\mathbb{R}^n)} + \overbrace{u}^{\in S(\mathbb{R}^n)} = \underbrace{f}_{\in S(\mathbb{R}^n)}$$

$$-\widehat{\Delta u}(\xi) + \hat{u}(\xi) = \hat{f}(\xi)$$

$$\widehat{\Delta u}(\xi) = \widehat{\left(\sum_{j=1}^{n} \partial^2_{x_j} u\right)}(\xi) =$$

$$= \sum_{j=1}^{n} \left(\partial^2_{x_j} u\right)^\wedge(\xi) = \sum_{j=1}^{n} (2\pi i \xi_j)^2 \, \hat{u}(\xi)$$

Kaida poch. $\partial_{x_j} \rightsquigarrow (2\pi i \xi_j)$

$$= -4\pi^2 \hat{u}(\xi) \sum_{j=1}^{n} \xi_j^2$$

$$= -4\pi^2 |\xi|^2 \hat{u}(\xi)$$

$$4\pi^2|\xi|^2\,\underbrace{\hat{u}(\xi)} + \underbrace{\hat{u}(\xi)} = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{4\pi^2|\xi|^2+1}$$

$$u(\xi) := \left(\frac{\hat{f}(\xi)}{4\pi^2|\xi|^2+1}\right)^{\vee}$$

$$f \in S(\mathbb{R}^d) \Downarrow \hat{f} \in S(\mathbb{R}^d)$$

$$S(\mathbb{R}^d) \ni \left(\frac{\hat{f}(\xi)}{4\pi|\xi|^2+1}\right)^{\vee} \Leftarrow S(\mathbb{R}^d) \ni \underline{\frac{\hat{f}(\xi)}{4\pi^2|\xi|^2+1}}$$

To co jest ważne to że $4\pi|\xi|^2 + 1 \geq 1$

Nasz kandydat na rozwiązanie:

$$u(\xi) = \left(\frac{\hat{f}(\xi)}{4\pi|\xi|^2 + 1} \right)^{\vee}.$$

u spełnia równanie $-\Delta u + u = f$ bo można to rozumowanie odwrócić.

Dlaczego jest tylko jedno takie u ?

Zał. że istnieją $u_1, u_2 \in S(\mathbb{R}^n)$

$$\begin{cases} -\Delta u_1 + u_1 = f \\ -\Delta u_2 + u_2 = f \end{cases}$$

$$-\Delta(u_1 - u_2) + u_1 - u_2 = 0$$

$u_1 - u_2 \in S(\mathbb{R}^n)$ $\qquad -\Delta \tilde{u} + \tilde{u} = 0$

Step 1

$$(u_1 - u_2)(x) = \left(\frac{0}{1 + 4\pi^2 |\xi|^2} \right)^{\vee} = 0.$$

Grafakos, Fourier analysis.

5) $f = u + \partial_1^2 \partial_2^2 \partial_3^4 u + 4i\,\partial_1^2 u + \partial_2^7 u$

$f \in S(\mathbb{R}^d)$ $\quad d = 3$

Znaleźć wszystkie $u \in S(\mathbb{R}^d)$ jak wyżej.

$(\partial_1^2 \partial_2^2 \partial_3^4 u)^\wedge(\xi) = (2\pi i)^8 \xi_1^2 \xi_2^2 \xi_3^4 \hat{u}(\xi)$

$(4i\,\partial_1^2 u)^\wedge(\xi) = 4i\,(2\pi i)^2 \xi_1^2 \hat{u}(\xi)$

$(\partial_2^7 u)^\wedge(\xi) = (2\pi i)^7 \xi_2^7 \hat{u}(\xi)$

$$\left(\partial_1^2 \partial_2^2 \partial_3^4 u\right)^\wedge(\xi) = (2\pi i)^8 \xi_1^2 \xi_2^2 \xi_3^4 \hat{u}(\xi)$$

$$\left(4i\partial_1^2 u\right)^\wedge(\xi) = 4i\,(2\pi i)^2 \xi_1^2 \hat{u}(\xi)$$

$$\left(\partial_2^7 u\right)^\wedge(\xi) = (2\pi i)^7 \xi_2^7 \hat{u}(\xi)$$

$$\hat{f} = \hat{u}(\xi)\left[1 + (2\pi)^8 \xi_1^2 \xi_2^2 \xi_3^4 + \underline{(-1)\, i\, 4\, (2\pi)^2 \xi_1^2} \right.$$

$$\left. \underline{-i\,(2\pi)^7 \xi_2^2} \right]$$

$$u(\xi) = \left(\frac{\hat{f}(\xi)}{\underbrace{1 + (2\pi)^8 \xi_1^2 \xi_2^2 \xi_3^4}_{\text{CZĘŚĆ RZECZ.} \geq 1 > 0} + i(\ldots)} \right)^{\vee}$$

Puenta: $f \in S(\mathbb{R}^d)$

$$\frac{f(x)}{p(x)} \in S(\mathbb{R}^d) \quad \text{o ile} \quad |p(x)| \geq c > 0.$$

- Równanie różniczkowe cząstkowe (Wstęp do...)
- Wybrane zagadnienia AF ...
- Elementy analizy rzeczywistej

  (AM II.2 zamiast form różniczkowych).

Szereg fouriera $\quad L^2(0,1)$ nad $\mathbb{C}$

$\{e^{2\pi i k x}\}_{k\in\mathbb{Z}}$ baza o.n. $L^2(0,1)$ nad $\mathbb{C}$

$$a_k = \langle f, e^{2\pi i k x}\rangle$$

$\sin(kx) + i\cos(kx)$.

$$f = \sum a_k e^{2\pi i k x} = \sum \langle f, e^{2\pi i k x}\rangle e^{2\pi i k x}$$

w $L^2(0,1)$

PerEnflo H – ośr. p. Hilberta

$$u = \sum_{i=1}^{\infty} \langle u, e_i \rangle e_i$$ szereg zbiega w H.

Pyt: E – ośr. p. Banacha $\exists \{e_i\}$

$$u = \sum_{i=1}^{\infty} \varphi_i(u)\, e_i \qquad \varphi_i \in E^*$$

PerEnflo: nie: