How to compute distance, projection, ... using orthonormal basis?

Let H be Hilbert space and $X = \text{span}\{w_1, \dots, w_n\}$, $X \subset H$. Assume that $\{w_i\}_{i=1}^n$ is an orthonormal basis of X i.e.

$$\langle w_i, w_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(1) PROJECTION:

$$P_X y = \sum_{i=1}^{n} \langle y, w_i \rangle w_i$$

PROOF OF (1): $P_X y \in X$ so there are $a_1, \dots, a_n$ s.t. $P_X y = \sum a_i w_i$. Moreover, $y - P_X y \perp P_X y$

so $y - \sum_{i=1}^{n} a_i w_i \perp w_j \quad \forall j = 1, \dots, n$

$$\Rightarrow \langle y, w_j \rangle = a_j \langle w_j, w_j \rangle = a_j.$$

$\square$.

Note: we used here that $\|w_i\| = 1$ (this is not true otherwise).

(2) DISTANCE

$$\mathrm{dist}^2(y, X) = \|y - P_X y\|^2 =$$

$$= \|y\|^2 - \sum_{i=1}^{n} \langle y, w_i \rangle^2.$$

PROOF: Note that $\|P_X y\|^2 = \sum_{i=1}^{n} \langle y, w_i \rangle^2$.

As $y - P_X y \perp P_X y$ we use Pythagorean theorem

$$\|y\|^2 = \|P_X y\|^2 + \|y - P_X y\|^2 \Rightarrow$$

$$\Rightarrow \|y - P_X y\|^2 = \|y\|^2 - \|P_X y\|^2 =$$

$$= \|y\|^2 - \sum_{i=1}^{n} \langle y, w_i \rangle^2.$$

$\square$.