Functional Analysis - HW 11

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1 Problem 1

1.1 Description

The following exercise shows that any compact and nonempty $K \subset \mathbb{C}$ is a spectrum of some $T : l^2 \to l^2$.

(A) Let $y \in l^\infty$ be a complex sequence and $T : l^2 \to l^2$ be defined with $Tx = (x_i y_i)_{i \in \mathbb{N}}$. Prove that $\sigma(T) = \overline{\{y_i : i \in \mathbb{N}\}}$ where the line above denotes closure of the set.

(B) Let $K \subset \mathbb{C}$ be a nonempty and compact subset. Construct a bounded linear operator $T : l^2 \to l^2$ such that $\sigma(T) = K$.

1.2 Solution

(A) First of all, let us see that $T - y_i I$ has nontrivial kernel for every $i \in \mathbb{N}$. Indeed, in this case we have $Te_i = y_i e_i$ (by e_i we denote a sequence with only zeros apart from position i where we put one). Therefore $e_i \in \ker T - y_i I$ and $y_i \in \sigma(T)$. As $\sigma(T)$ is a closed set, then we have $\overline{\{y_i : i \in \mathbb{N}\}} \subseteq \sigma(T)$.

Now let us take $z \in \mathbb{C}$ such that there is an $\varepsilon > 0$ for which $B(z, \varepsilon) \cap \{y_i : i \in \mathbb{N}\} = \emptyset$. We will show $T - zI$ is reversible. Indeed, in this case $(T - zI)(x) = ((y_i - z)x_i)_{i \in \mathbb{N}}$, so we can write $(T - zI)^{-1}(x) = (x_i/(y_i - z))_{i \in \mathbb{N}}$. Obviously this is a proper inversion. Moreover, using our assumption on z, we have $|y_i - z| \geq \varepsilon$, so the bound $\left\|(T - zI)^{-1}\right\| \leq 1/\varepsilon$ easily follows. Therefore $z \notin \sigma(T)$, as there exists a bounded inverse of $T - zI$. It follows that $\sigma(T) \subseteq \overline{\{y_i : i \in \mathbb{N}\}}$.

As we have both inclusions, we conclude $\sigma(T) = \overline{\{y_i : i \in \mathbb{N}\}}$.

(B) Let us take $\widehat{K} = K \cap \{x + iy : (x, y) \in \mathbb{Q}^2\}$. Obviously $\overline{\widehat{K}} = K$ and $\widehat{K}$ is at most countable, so we can arbitrary order its elements obtaining a sequence $y = (y_1, y_2, \ldots)$. In the case $\widehat{K}$ is finite, we repeat the last element infinitely many times. As K is compact, it is bounded and the same holds for $\widehat{K}$ and $y \in l^\infty$. Therefore we can use item **(A)** and construct an operator T such that $\sigma(T) = \overline{\widehat{K}} = K$.

2 Problem 2

2.1 Description

For the following operators briefly justify that they are bounded and linear. Find their adjoints.

(A) $K : L^2(0, 1) \to L^2(0, 1)$ defined with $Kf(x) = \int_0^x f(y)\,\mathrm{d}y$.

(B) $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined with $Tf(x) = \mathrm{sgn}(x) f(x + 1)$.

2.2 Solution

(A) Using linearity of integral we get

$$\begin{aligned} K(f+g)(x) &= \int_0^x (f+g)(y)\,\mathrm{d}y \\ &= \int_0^x f(y) + g(y)\,\mathrm{d}y \\ &= \int_0^x f(y)\,\mathrm{d}y + \int_0^x g(y)\,\mathrm{d}y \\ &= Kf(x) + Kg(x), \end{aligned}$$

so the operator is linear. To show that K is bounded we estimate

$$\begin{aligned}
\|Kf\|_2^2 &= \int_0^1 \left(\int_0^x f(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\
&= \int_0^1 \left(\int_0^1 \mathbb{1}_{(0,x)}(y) f(y)\,\mathrm{d}y\right)^2 \mathrm{d}x \\
&= \int_0^1 \left|\left\langle f, \mathbb{1}_{(0,x)}\right\rangle\right|^2 \mathrm{d}x \\
&\le \int_0^1 \|f\|_2^2 \left\|\mathbb{1}_{(0,x)}\right\|_2^2 \mathrm{d}x \\
&= \|f\|_2^2 \int_0^1 \left\|\mathbb{1}_{(0,x)}\right\|_2^2 \mathrm{d}x \\
&= \|f\|_2^2 \int_0^1 \left\|\mathbb{1}_{(0,x)}\right\|_2^2 \mathrm{d}x \\
&= \|f\|_2^2 \int_0^1 x\,\mathrm{d}x \\
&= \frac{1}{2} \|f\|_2^2 ,
\end{aligned}$$

where we used Cauchy-Schwarz inequality. We'll now determine the adjoint of K. Let $g \in L^2(0,1)$. We can compute

$$\begin{aligned}
\langle Kf, g\rangle &= \int_0^1 \left(\int_0^x f(y)\,\mathrm{d}y\right) \overline{g(x)}\,\mathrm{d}x \\
&= \int_0^1 \left(\int_0^1 \mathbb{1}_A(x,y) f(y)\,\mathrm{d}y\right) \overline{g(x)}\,\mathrm{d}x \\
&= \int_{(0,1)^2} \mathbb{1}_A(x,y) f(y)\overline{g(x)}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{(0,1)^2} \mathbb{1}_A(x,y) f(y)\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left(\int_0^1 \mathbb{1}_A(x,y)\overline{g(x)}\,\mathrm{d}x\right) f(y)\,\mathrm{d}y \\
&= \int_0^1 f(y) \left(\int_y^1 \overline{g(x)}\,\mathrm{d}x\right) \mathrm{d}y \\
&= \int_0^1 f(y) \overline{\left(\int_y^1 g(x)\,\mathrm{d}x\right)}\,\mathrm{d}y \\
&= \langle f, Sg\rangle ,
\end{aligned}$$

where we defined $A = \{(x,y) : 0 < y < x < 1\}$, we used Fubini theorem and the fact that when integrating over subset of $\mathbb{R}$ the integral of function's complex conjugate is equal to the conjugate of the integral. We conclude that the adjoint operator T^* is equal to S, so

$$T^* f(x) = \int_x^1 f(y) \, \mathrm{d}y.$$

(B) The operator is linear because

$$\begin{aligned} K(f+g)(x) &= \operatorname{sgn}(x)((f+g)(x+1)) \\ &= \operatorname{sgn}(x) f(x+1) + \operatorname{sgn}(x) g(x+1) \\ &= Kf(x) + Kg(x). \end{aligned}$$

It is also bounded as

$$\begin{aligned} \|Kf\|_2^2 &= \int_{\mathbb{R}} Kf(x)^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}} (\operatorname{sgn}(x) f(x+1))^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}} f(x+1)^2 \, \mathrm{d}x \\ &= \|f\|_2^2 , \end{aligned}$$

where we used the definition of sign function and the fact that $\{0\}$ is a set of measure 0. Now we can compute

$$\begin{aligned} \langle Kf, g \rangle &= \int_{\mathbb{R}} Kf(x) \overline{g(x)} \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \operatorname{sgn}(x) f(x+1) \overline{g(x)} \, \mathrm{d}x \\ &= \int_{\mathbb{R}} f(t) \operatorname{sgn}(t-1) \overline{g(t-1)} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} f(t) \overline{\operatorname{sgn}(t-1) g(t-1)} \, \mathrm{d}t \\ &= \langle f, Sg \rangle , \end{aligned}$$

so the adjoint K^* is equal to S, so

$$K^* f(x) = \operatorname{sgn}(x-1) f(x-1).$$