Functional Analysis - HW 3

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1 Problem 1

1.1 Description

Let $1 \le p \le \infty$. Compute norms of the operators

a. $T : l^p \to l^p$ defined with $T((a_n)_{n\ge1}) = (a_{n+1} - a_n)_{n\ge1}$.

b. $T : L^p(0,1) \to L^p(0,1)$ defined with $(Tf)(x) = f(\sqrt{x})$.

1.2 Solution

a. Suppose that $p = \infty$. Take $a = (a_1, a_2, \dots) \in l^\infty$ such that $\|a\|_\infty = 1$. Then

$$\|T(a)\|_\infty = \sup_{n\in\mathbb{N}} |a_{n+1} - a_n| \le 2 \sup_{n\in\mathbb{N}} |a_n| = 2\|a\|_\infty = 2.$$

Additionally, for $a = (1, -1, 1, -1, \dots)$ we have $\|a\|_\infty = 1$ and $\|T(a)\|_\infty = 2$, because $T(a) = (-2, 2, -2, 2, \dots)$, so we conclude that $\|T\| = 2$.

Now consider $1 \le p < \infty$. Take $a = (a_1, a_2, \dots) \in l^p$ such that $\|a\|_p = 1$. Using Minkowski inequality we obtain

$$\begin{aligned}\|T(a)\|_p &= \left(\sum_{n=1}^{\infty} |a_{n+1} - a_n|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{n=1}^{\infty} |a_{n+1}|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \\ &= (\|a\|_p^p - |a_1|)^{\frac{1}{p}} + \|a\|_p \\ &= (1 - |a_1|)^{\frac{1}{p}} + 1.\end{aligned}$$

We know that $\|a\|_p = 1$, so $0 \leq |a_1| \leq 1$, hence

$$\|T(a)\|_p \leq (1 - |a_1|)^{\frac{1}{p}} + 1 \leq 2.$$

We'll show that 2 is indeed the norm of T. Consider $a_\lambda = (t, \lambda t, \lambda^2 t, \dots)$ where $0 > \lambda > -1$ and t is chosen so that $t^p + |\lambda|^p = 1$. We check that

$$\begin{aligned}
\|a_\lambda\|_p &= \left(\sum_{n=0}^{\infty} |\lambda^n t|^p \right)^{\frac{1}{p}} \\
&= \left(t^p \sum_{n=0}^{\infty} |\lambda|^{np} \right)^{\frac{1}{p}} \\
&= \left(t^p \frac{1}{1 - |\lambda|^p} \right)^{\frac{1}{p}} \\
&= 1.
\end{aligned}$$

Let's compute the norm of $T(a_\lambda)$:

$$\begin{aligned}
\|T(a_\lambda)\|_p &= \left(\sum_{n=0}^{\infty} |\lambda^{n+1} t - \lambda^n t|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{n=0}^{\infty} |\lambda^n t(\lambda - 1)|^p \right)^{\frac{1}{p}} \\
&= \left(t^p (1 - \lambda)^p \sum_{n=0}^{\infty} |\lambda|^{np} \right)^{\frac{1}{p}} \\
&= \left(t^p (1 - \lambda)^p \frac{1}{1 - |\lambda|^p} \right)^{\frac{1}{p}} \\
&= 1 - \lambda.
\end{aligned}$$

Hence, $\lim_{\lambda \to (-1)} \|T(a_\lambda)\|_p = \lim_{\lambda \to (-1)} (1 - \lambda) = 2$, so $\|T\| = 2$.

b. Suppose $p = \infty$. Let $\varphi : (0,1) \to (0,1)$ be the map that is bijective and differentiable. Take measurable $S \subset (0,1)$. Using the change of variables formula for $\mathbb{1}_{\varphi(S)}$ we obtain

$$\mu(\varphi(S)) = \int_S |\det \mathrm{D}\varphi| \, \mathrm{d}\mu,$$

which implies

$$\mu(\varphi(S)) = 0 \iff \varphi(S) = 0.$$

Now let $\phi : (0,1) \to (0,1)$ be defined as $\phi(x) = x^2$. It is indeed bijective and differentiable. Analogously for its inverse $\phi^{-1}(x) = \sqrt{x}$. Let $f \in L^\infty$. For $a \in \mathbb{R}$ we have

$$\mu(|f|^{-1}((a,\infty))) = 0 \iff \mu(\phi(|f|^{-1}((a,\infty)))) = 0,$$

but $\mu((|f| \circ \phi^{-1})^{-1}((a,\infty))) = \mu((|Tf|)^{-1}((a,\infty)))$, which implies that

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \operatorname{ess\,sup} |Tf| = \|Tf\|_\infty .$$

That implies that $\|T\| = 1$ and we are done.

Now consider $1 \leq p < \infty$. Take $f \in L^p(0,1)$ such that $\|f\|_p = 1$. We see that

$$\|Tf\|_p^p = \int_0^1 |Tf|^p \, \mathrm{d}\mu,$$

after which we can use the change of variables formula (ϕ satisfies conditions) and compute

$$\begin{aligned}\int_0^1 |Tf|^p \, \mathrm{d}\mu &= \int_0^1 (|Tf|^p \circ \phi) \cdot |\det \mathrm{D}\phi| \, \mathrm{d}\mu \\ &= \int_0^1 |f|^p \cdot |2x| \, \mathrm{d}\mu \\ &\leq \|f\|_p^p \sup_{x \in (0,1)} |2x| \\ &= 2.\end{aligned}$$

Hence, $\|Tf\|_p \leq 2^{\frac{1}{p}}$, so $\|T\| \leq 2^{\frac{1}{p}}$. We'll show that indeed $\|T\| = 2^{\frac{1}{p}}$. Let $f_t = \left(\frac{1}{1-t}\right)^{\frac{1}{p}} \mathbb{1}_{(t,1)}$ for $0 < t < 1$. We check that

$$\begin{aligned}\|f_t\|_p^p &= \int_0^1 \left| \left(\frac{1}{1-t}\right)^{\frac{1}{p}} \mathbb{1}_{(t,1)} \right|^p \mathrm{d}\mu \\ &= \frac{1}{1-t} \int_t^1 \mathrm{d}\mu \\ &= 1.\end{aligned}$$

Let's estimate the p-th power of the norm of Tf_t:

$$\begin{aligned}
\|Tf_t\|_p^p &= \int_0^1 |f_t|^p \cdot |2x| \, \mathrm{d}\mu \\
&= \int_0^1 \frac{1}{1-t} \mathbb{1}_{(t,1)} \cdot |2x| \, \mathrm{d}\mu \\
&= \int_t^1 \frac{1}{1-t} \cdot |2x| \, \mathrm{d}\mu \\
&\geq \int_t^1 \frac{1}{1-t} \, \mathrm{d}\mu \cdot \inf_{x \in (t,1)} |2x| \\
&= 2t.
\end{aligned}$$

Hence, for every $0 < t < 1$ the inequalities $(2t)^{\frac{1}{p}} \leq \|Tf_t\|_p \leq 2^{\frac{1}{p}}$ hold. That means $\lim_{t \to 1} \|Tf_t\|_p$ exists and it is equal to $2^{\frac{1}{p}}$, so $\|T\| = 2^{\frac{1}{p}}$.

2 Problem 2

2.1 Description

Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Suppose that D is a dense linear subset of X and $T : (D, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a bounded linear operator. Prove that T has a unique bounded extension

$$\widetilde{T} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

such that

$$Tx = \widetilde{T}x \text{ for } x \in D \text{ and } \|T\|_{\mathcal{L}(D,Y)} = \left\|\widetilde{T}\right\|_{\mathcal{L}(X,Y)}.$$

2.2 Solution

We will start with proving existence of such operator. As D is dense in X then for every $x \in X$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \to x$. Let us denote $y_n = T(x_n)$. It is straightforward to prove that (y_n) is a Cauchy sequence in Y. Indeed, x_n converges so it is a Cauchy sequence. Moreover $\|y_n - y_m\|_Y = \|T(x_n - x_m)\|_Y \leq \|T\| \, \|x_n - x_m\|_X$, so the Cauchy condition follows. As $(Y, \|\cdot\|_Y)$ is Banach, then $y_n \to y \in Y$. We can set $\widetilde{T}(x) = y$.

We need to prove that this definition of $\widetilde{T}$ is sound. Assume that there is $x \in X$ and two different sequences $(x_n)_{n \in \mathbb{N}}$ and $(\widehat{x}_n)_{n \in \mathbb{N}}$ such that both $x_n \to x$ and $\widehat{x}_n \to x$. We need to prove that $\lim T(x_n) = \lim T(\widehat{x}_n)$. We clearly have that $x_1, \widehat{x}_1, x_2, \widehat{x}_2, \ldots$ is a sequence in X that converges to x. Therefore, as in the previous paragraph, the sequence $T(x_1), T(\widehat{x}_1), T(x_2), T(\widehat{x}_2), \ldots$ converges in Y. Therefore every subsequence of it converges to the same element of Y. Therefore we have $\lim T(x_n) = \lim T(\widehat{x}_n)$ and we know that this definition is sound.

Obviously for any $x \in D$ to compute $\widetilde{T}(x)$ we can use the constant sequence $(x)_{n\in N}$ and we get $\widetilde{T}(x) = T(x)$.

The last thing in this part is to prove that $\widetilde{T}$ is bounded. Let us take any $x \in X$ such that $\|x\|_X = 1$. Again, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of D such that $x_n \to x$. We know that $T(x_n) \to \widetilde{T}(x)$. Hence, by the definition of convergence, we get $\left\|T(x_n) - \widetilde{T}(x)\right\|_Y \to 0$. By the triangle inequality we get $\left\|\widetilde{T}(x)\right\|_Y \leq \left\|T(x_n) - \widetilde{T}(x)\right\|_Y + \|T(x_n)\|_Y$. This gives us $\left\|\widetilde{T}\right\| \leq \|T\|$. Obviously $\left\|\widetilde{T}\right\| \geq \|T\|$ (as $\widetilde{T}$ is an extension of T), so $\left\|\widetilde{T}\right\| = \|T\|$ as we wanted.

No we can prove the second part, namely the uniqueness of $\widetilde{T}$. This is much easier. Indeed, if there is any other extension $\widehat{T}$ then it is continuous, so for any $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D$ and $x_n \to x$ we have $\widehat{T}(x_n) \to \widehat{T}(x)$. As $\widehat{T}(x_n) = T(x_n)$ then we have $T(x_n) \to \widehat{T}(x)$. But $T(x_n) \to \widetilde{T}(x)$ by definition of $\widetilde{T}$ so $\widetilde{T}(x) = \widehat{T}(x)$ for any $x \in X$.