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First problem

Let us assume that $(L^p(0,1), \|\cdot\|_q)$ is a Banach space. We know that $(L^p(0,1), \|\cdot\|_p)$ is a Banach space and from B1/PS1 we have $\|f\|_q \leq \|f\|_p$ for all $f \in L^p(0,1)$ (the Lebesgue measure of $(0,1)$ equals 1). Hence from B1/PS4 we obtain that exists $C \in \mathbb{R}$ such that $\|f\|_p \leq C\|f\|_q$ for all $f \in L^p(0,1)$.

We will show that such C can not exist. Let us consider a sequence $(f_n) \subset L^p(0,1)$ such that $f_n = \mathbb{1}_{(1/n,1)} x^{-1/p}$. Then for $n = 1, 2, \ldots$ we have

$$\|f_n\|_p = \left(\int_0^1 \mathbb{1}_{(1/n,1)} x^{-1} \mathrm{d}x\right)^{1/p}$$

$$\|f_n\|_q = \left(\int_0^1 \mathbb{1}_{(1/n,1)} x^{-q/p} \mathrm{d}x\right)^{1/q} \leq \left(\int_0^1 x^{-q/p} \mathrm{d}x\right)^{1/q} = \left(\frac{1}{1-q/p}\right)^{1/q} =: a \in \mathbb{R}$$

because $q/p < 1$. Hence $\left(\int_{1/n}^1 x^{-1}\mathrm{d}x\right)^{1/p} \leq C_1$ (where $C_1 = aC$) what leads to a contradiction, because $\int_{1/n}^1 x^{-1}\mathrm{d}x \to \infty$ in $\mathbb{R}$ for $n \to \infty$.

Second problem

For a Banach space X and an operator $P : X \to X$ such that $\ker(P)$, $\mathrm{im}(P)$ are closed in X and $\forall x \in X\ P(Px) = Px$ let us observe firstly that P limited to $\mathrm{im}(P)$ is the identity operator. Because, by definition of image $\forall b \in \mathrm{im}(P)\ \exists a \in X\ Pb = P(Pa) = Pa = b$ following from the last assumption.

Now we will show that the graph $G(P)$ is closed in $X \times X$. Let us consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x \in X$ and $Px_n \to y \in X$. Then $y \in \mathrm{im}(P)$, since the image is closed, so $Py = y$. Furthermore, $x_n - Px_n \in \ker(P)$, because $P(x_n - Px_n) = Px_n - Px_n = 0$, thus by closedness of $\ker(P)$ we obtain that $x - y$ also belongs to the kernel. Hence, $P(x-y) = 0$, i.e. $Px = Py = y$, which means that $G(P)$ is closed.

Therefore, we can apply Closed Graph Theorem™, what gives us $P \in \mathcal{L}(X, X)$.

Third problem

Let us observe that $E = \{f \in L^2(-1,1) : \int_{-1}^1 f(t)\mathrm{d}t = \int_{-1}^1 f(t)t\mathrm{d}t\} = \{f \in L^2(-1,1) : \langle f, 1\rangle = \langle f, t\rangle = 0\}$. Hence for $K = \operatorname{span}\{1, t\}$ we obtain $E = K^\perp$ (i.e. from B1/PS5 E is a closed subspace of $L^2(-1,1)$). Because K is a closed subspace of $L^2(-1,1)$ (it is a finite-dimensional subspace), $L^2(-1,1) = E \oplus K$.

We have $g = P_E g + P_K g$. Because $P_K g \in K$ we have $P_K g = a + bt$ for some $a, b \in \mathbb{R}$. We can calculate a and b from: $g - P_K g = P_E g \in E \iff (P_E g \perp 1) \wedge (P_E g \perp t)$.

$$g - P_K g \perp 1 \iff 0 = \langle g - P_K g, 1\rangle = \int_{-1}^1 (g - P_K g)\mathrm{d}t = \int_{-1}^1 \left(\frac{1}{1+t^2} - a - bt\right)\mathrm{d}t = \frac{\pi}{2} - 2a$$

$$g - P_K g \perp t \iff 0 = \langle g - P_K g, t\rangle = \int_{-1}^1 (g - P_K g)t\mathrm{d}t = \int_{-1}^1 \left(\frac{t}{1+t^2} - at - bt^2\right)\mathrm{d}t = -\frac{2}{3}b$$

Hence $a = \frac{\pi}{4}$ and $b = 0$ and we have

$$P_{E^\perp} g = P_K g = \frac{\pi}{4}$$
$$P_E g = g - P_K g = \frac{1}{1+t^2} - \frac{\pi}{4}.$$

Now we can calculate the distance $\operatorname{dist}(g, E)$ of g from E.

$$\operatorname{dist}(g, E) = \left(\int_{-1}^1 |g - P_E g|^2 \mathrm{d}t\right)^{1/2} = \left(\int_{-1}^1 |P_K g|^2 \mathrm{d}t\right)^{1/2} = \left(\int_{-1}^1 \left|\frac{\pi}{4}\right|^2 \mathrm{d}t\right)^{1/2} = 2^{-3/2}\pi.$$