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Problem 2

(A) Let us take any f satisfying assumptions and denote the maximum of $|f|$ by M. We want to prove

$$\lim_{n\to\infty}\int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)dx\int_0^1 g(x)dx$$

for any $g \in L^2(0,1)$. We will start with a special case, when g is an indicator, i.e. $g = \mathbb{1}_{(a,b)}$ for some $0 \leq a < b \leq 1$. In that case

$$\lim_{n\to\infty}\int_0^1 f_n(x)g(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx.$$

Assuming that there are some $a' = \frac{k}{n}, b' = \frac{l}{n}$ for some integers k, l such that $a \leq a' \leq a + \frac{1}{n}$ and $b \leq b' \leq b + \frac{1}{n}$, we can estimate:

$$\left|\int_a^b f_n(x)dx - (b-a)\int_0^1 f(x)dx\right| \leq$$

$$\leq \int_a^{a'} |f_n(x)|dx + \left|\int_{a'}^{b'} f_n(x) - f_n(x)dx\right| + \int_b^{b'} |f_n(x)|dx + \frac{2}{n}\int_0^1 |f(x)|dx \leq \frac{4M}{n},$$

which is because $(b'-a')\int_0^1 f(x)dx = \int_{a'}^{b'} f_n(x)dx$ (and this can be easily seen by the change of variable theorem). As M is constant we get the statement for this special case of g being an interval indicator.

Obviously the statement follows as well for g being a simple function, i.e. a finite sum of intervals' indicators. Now we can proceed to the general case, which is essentially the same as something we did during tutorials. Indeed, let g be a function in $L^2(0,1)$ and fix some $\varepsilon > 0$. There is a simple function g_ε such that $\|g - g_\varepsilon\|_2 \leq \varepsilon$. For any n we can estimate:

$$\left|\int_0^1 f_n(x)g(x)dx\right| = \left|\int_0^1 f_n(x)g_\varepsilon(x)dx\right| + \left|\int_0^1 f_n(x)(g-g_\varepsilon)(x)dx\right| \leq$$

$$\leq \left|\int_0^1 f_n(x)g_\varepsilon(x)dx\right| + \|g - g_\varepsilon\|_2 \|f_n\|_2 \leq \left|\int_0^1 f_n(x)g_\varepsilon(x)dx\right| + \varepsilon M,$$

where we used Hölder inequality between the first and the second line. To finish the proof, we can apply lim sup to both sides, thus obtaining:

$$\limsup_{n\to\infty}\left|\int_0^1 f_n(x)g(x)dx\right| \leq \varepsilon M.$$

As ε can be arbitrarily small, the statement follows.

(B) Using **(A)** we obtain that:

$$\sin(2\pi nx) \rightharpoonup \int_0^1 \sin(2\pi x)dx = \frac{-\cos(2\pi x)}{2\pi}\bigg|_0^1 = \frac{-1}{2\pi}(\cos(2\pi) - \cos(0)) = 0,$$

and:

$$\sin^2(2\pi nx) \rightharpoonup \int_0^1 \sin^2(2\pi x)dx = \int_0^1 \frac{1-\cos(4\pi x)}{2}dx =$$

$$= \frac{1}{2} - \frac{1}{2}\int_0^1 \cos(4\pi x)dx = \frac{1}{2} - \frac{\sin(4\pi x)}{8\pi}\bigg|_0^1 = \frac{1}{2} - \frac{1}{8\pi}(\sin(4\pi) - \sin(0)) = \frac{1}{2}.$$