Functional Analysis, PS10

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A4 $(T_1+T_2)^* = T_1^* + T_2^*$

$$\langle (T_1+T_2)x, y\rangle = \langle T_1 x, y\rangle + \langle T_2 x, y\rangle =$$

$$= \langle x, T_1^* y\rangle + \langle x, T_2^* y\rangle = \langle x, (T_1^*+T_2^*)y\rangle.$$

A5 $(\lambda T)^* = \bar{\lambda} T^*$

$$\langle \lambda T x, y\rangle = \langle \lambda x, T^* y\rangle = \lambda \langle x, T^* y\rangle =$$

$$= \langle x, \bar{\lambda} T^* y\rangle.$$

B1

$$\langle x, y \rangle = x^T \cdot \overline{y}$$

$$\langle Ax, y \rangle = (Ax)^T \overline{y} = x^T A^T \overline{y} = \langle x, \overline{A}^T y \rangle$$

$$\Rightarrow A^* = \overline{A}^T.$$

B2

$$(Rx)_k = x_{k-1} \qquad \left(R : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})\right).$$

$$(Lx)_k = x_{k+1}$$

$$\|R\| = 1. \qquad \|L\| = 1$$

$$R^{-1} = L \qquad L^{-1} = R$$

$$\langle Rx, y \rangle = \sum_k (Rx)_k y_k = \sum_k x_{k-1} y_k =$$

$$= \sum_k x_{k-1} (Ly)_{k-1} = \langle x, Ly \rangle \Rightarrow R^* = L$$

Using involution property $L^* = R$.

B4 $P_M : H \to H$ $\quad x, y \in H$ $\quad x = x_M + x_{M^\perp}$

$y = y_M + y_{M^\perp}$.

$$\langle P_M x, y \rangle = \langle x_M, y \rangle$$

$$= \langle x_M, y_M + y_{M^\perp} \rangle = \langle x_M, y_M \rangle$$

$$= \langle x_{M^\perp} + x_M, y_M \rangle = \langle x, P_M y \rangle.$$

$$\Rightarrow (P_M)^* = P_M.$$

B5 $$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad (\text{limit in } L(H,H))$$

$$(A^k)^* = (A^*)^k \quad \forall k$$

Hence, for finite sum

$$\left\langle \left(\sum_{k=0}^{N} \frac{A^k}{k!} \right) x, y \right\rangle = \left\langle x, \left(\sum_{k=0}^{N} \frac{{A^*}^k}{k!} \right) y \right\rangle$$

Pass to the limit with $k \to \infty$

$$\langle e^A x, y \rangle = \langle x, e^{A^*} y \rangle.$$

B6 $Tf(x) = \int_0^1 K(x,y)\, f(y)\, dy$

$$\langle Tf, g\rangle = \int_0^1 Tf(x)\, \overline{g(x)}\, dx =$$

$$= \int_0^1 \int_0^1 K(x,y)\, f(y)\, \overline{g(x)}\, dy\, dx \overset{\uparrow}{\underset{\text{Fubini}}{=}}$$

$$= \int_0^1 \left[\int_0^1 K(x,y)\, \overline{g(x)}\, dx \right] f(y)\, dy$$

$$= \int_0^1 f(y)\, \overline{\int_0^1 \overline{K(x,y)}\, g(x)\, dx}\, dy$$

$$= \langle f, K^* g\rangle$$

$$K^* g(y) = \int_0^1 \overline{K(x,y)}\, g(x)\, dx$$

C3 $(P_M)^* = P_M \Rightarrow P_M$ is self-adjoint

$\sigma(P_M) = ?$

$$(P_M - \lambda I) = \begin{cases} P_M & \lambda = 0 \\ P_{M^\perp} & \lambda = 1 \end{cases} \Rightarrow \text{not invertible}$$

For $\lambda \neq \{0,1\}$

- injectivity $(P_M - \lambda I)x = 0 \Rightarrow P_M x = \lambda x$

$$P_M x = \lambda P_M x \Rightarrow P_M x = 0 \qquad \Downarrow \qquad P_{M^\perp} x = 0$$

- surjectivity. Fix $y \in H$, find $x \in H$ s.t.

$$(P_M - \lambda I)x = y$$

$\hookrightarrow P_M x - \lambda P_M x = P_M y$. Take $y = P_M y + P_{M^\perp} y$.

$\hookrightarrow -\lambda P_{M^\perp} x = P_{M^\perp} y$

$\sigma(P_M)$ is purely residual.

C4 $M: L^2(0,1) \to L^2(0,1) \quad Mf(x) = xf(x)$

$\sigma(M) = [0,1]$.

$$\langle Mf, g \rangle = \int Mf(x)\,\overline{g(x)}\,dx = \int x\, f(x)\,\overline{g(x)}\,dx$$

$$= \int_0^1 f(x)\,\overline{x\,g(x)}\,dx = \langle f, Mg \rangle \quad \text{as } x \in \mathbb{R}.$$

C1 $\langle Tx, y \rangle = \langle x, Ty \rangle. \qquad T: H \to H$

$G(T) = \{ (x, Tx) \in H \times H \}$

Let $(x_n, Tx_n) \longrightarrow (x, y)$ in $H \times H$. We need $y = Tx$.

$$\langle Tx_n, z \rangle = \langle x_n, Tz \rangle \Rightarrow \langle y, z \rangle = \langle x, Tz \rangle = \langle Tx, z \rangle$$

$\Rightarrow y = Tx. \quad \checkmark$