Functional Analysis, PS11

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A1 $T: E \to F$ compact

$\Rightarrow \overline{T(B_1(0))}$ is compact

$\Rightarrow \sup_{\|x\| \leq 1} \|Tx\|_F \leq C \Rightarrow \|T\| \leq C.$

as compact sets are bounded.

A2 (A) $\overline{T(B_1(0))}$ is compact

(B) $\{x_n\}$ bdd $\Rightarrow \{Tx_n\}$ has conv. subseq.

(A) $\Rightarrow$ (B): $M = \sup_n \|x_n\|$, then $\frac{x_n}{M} \in B_1(0)$

$\Rightarrow \frac{T(x_n)}{M} \in \overline{T(B_1(0))} \Rightarrow \frac{T(x_{n_k})}{M} \to a$

$\Rightarrow T(x_{n_k}) \to a M.$

(B) $\Rightarrow$ (A):

Let $\{y_n\} \in \overline{T(B(0,1))}$. We want to find a subseq. y_{n_k} convergent in F.

Choose $z_n \in T(B(0,1))$, $\|z_n - y_n\| \leq \frac{1}{n}$.
By (B) z_n has a converging subsequence $z_{n_k} \longrightarrow z$. But then $y_{n_k} \longrightarrow z$ too since

$$\|y_{n_k} - z\| \leq \frac{1}{n_k} + \|z - z_{n_k}\| \longrightarrow 0$$

as $n_k \to \infty$.

A3 $T: E \to F$ $\quad S: E \to F$ compact
$\Rightarrow T+S$ is compact.

$\{x_n\}$ bdd in E. Choose subseq. s.t. $\{Tx_{n_k}\}$ converges and another one s.t. $\{Sx_{n_{k_l}}\}$ con.
It follows that $\{(T+S)x_{n_{k_l}}\}$ is convergent.

A4 $g \in C[0,1]$

$$T: C[0,1] \to C[0,1]$$

$$(Tf)(x) = \int_0^x f(y)\, g(y)\, dy$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be bdd in $C[0,1]$. We need to find subseq. $\{Tf_{n_k}\}$ converging in $C[0,1]$.

We use A-A:

(1) $|Tf_n(x)| \leq \|f_n\|_\infty \|g\|_\infty \leq \|g\|_\infty \sup_n \|f_n\|$
$\leq C.$

(2) $|(Tf_n)(x) - (Tf_n)(y)| = \int_x^y f_n(y)\, g(y)\, dy$

$\leq |y-x| \, \|f_n\|_\infty \|g\|_\infty \leq C\,|y-x|.$

$\square$.

A5 I is not compact as balls are not compact in ∞ dim HS.

A6 Otherwise I would be compact as

$$I = (I-T) + T.$$

A9 Hilbert-Schmidt operators $Tf(x) = \int_\Omega K(x,y) f(y)\,dy$

kernel

If K is continuous, it was discussed in the lecture that T is compact.

Now, we assume that $K \in L^2(\Omega \times \Omega)$. Note that this means that for a.e. $x \in \Omega$ $\int_\Omega K^2(x,y)\,dy < \infty$. (i.e. $K(x,\cdot) \in L^2(\Omega)$).

Again, let f_n be bdd in $L^2(\Omega)$. Choose a subsequence of $\{f_n\}$ converging weakly in $L^2(\Omega)$, by Banach-Alaoglu, i.e. $f_{n_k} \rightharpoonup f$.

As $K(x,\cdot) \in L^2(\Omega)$ for a.e. $x \in \Omega$; for a.e. $x \in \Omega$

$$\int_\Omega K(x,y) f_{n_k}(y)\,dy \longrightarrow \int_\Omega K(x,y) f(y)\,dy \quad (\text{a.e. } x \in \Omega).$$

So we obtained $Tf_{n_k} \longrightarrow Tf$ a.e. $x \in \Omega$. We need to upgrade this convergence to $L^2(\Omega)$. Note that

$$|Tf_{n_k}(x)| \leq \int_\Omega |K(x,y)|\,|f_{n_k}(y)|\,dy \overset{\text{Hölder}}{\leq} \|f_{n_k}\|_2 \, \|K(x,\cdot)\|_2 \leq$$

$$\leq C\,\|K(x,\cdot)\|_2 = C\left(\int_\Omega K^2(x,y)\,dy\right)^{1/2}$$

$$\Rightarrow |Tf_{n_k}(x)|^2 \leq \underbrace{C^2 \int_\Omega K^2(x,y)\,dy}$$ ← integrable majorant

$\Rightarrow$ $Tf_{n_k} \longrightarrow Tf$ in $L^2(\Omega)$ by dominated convergence. $\square$

B2 $(Tf)(x) = \int_0^x f(y)\,dy$

We first prove that $T: L^2(0,1) \to L^2(0,1)$ is compact. Let $\{f_n\}$ bdd in $L^2(0,1)$. We need to prove that $\{Tf_n\}$ has converging subsequence in $L^2(0,1)$. First, we find this subsequence in $C([0,1])$.

$$\|Tf_n\|_\infty \leq \int_0^1 |f_n(y)|\,dy \leq \left(\int_0^1 |f_n(y)|^2 dy\right)^{1/2}$$
$$\leq \|f_n\|_2 .$$

$$|Tf_n(x) - Tf_n(y)| = \int_x^y |f_n(z)|\,dz =$$

$$= \int_0^1 |f_n(z)| \, \mathbb{1}_{z \in [x,y]} \, dz \leq$$
$$\leq \left(\int_0^1 |f_n(z)|^2\right)^{1/2} \left(\int_0^1 \mathbb{1}^2_{z \in [x,y]} \, dz\right)^{1/2}$$

$$\leq \|f_n\|^2 \, |x-y|^{1/2} \leq C\,|x-y|^{1/2}$$

So there is subsequence converging wrt $\|\cdot\|_\infty$ norm so it converges also wrt $\|\cdot\|_2$ norm.

(we use here that $[0,1]$ has finite measure).

Now, when we know that T is compact, $0 \in \sigma(T)$ and any $\lambda \in \sigma(T)$, $\lambda \neq 0$ is its eigenvalue.

$$Tf = \lambda f \quad \text{i.e.} \quad \int_0^x f(y)\,dy = \lambda f(x)$$

It follows that f is C^1, $f(0) = 0$ and

$$\begin{cases} f'(x) = \frac{1}{\lambda} f(x). \\ f(0) = 0 \end{cases}$$

There is exactly one f solving this ODE: $\underline{f = 0}$.

$\square$.

B4 First, we find $\sigma(G)$

$$\sigma(G) = \overline{\{ g(x) : x \in \mathbb{R} \}}$$

$\supseteq$: If $\lambda = g(y)$ $\exists_{y \in \mathbb{R}}$ then $(G - \lambda I) f$

$$= (g(x) - \lambda) f(x) = (g(x) - g(y)) f(x)$$

g is continuous so $\forall_{\varepsilon > 0} \exists_{\delta_\varepsilon > 0} \forall_{x} |x-y| \leq \delta_\varepsilon$

$\Rightarrow |g(x) - g(y)| \leq \varepsilon$. y is fixed.

$$f_\varepsilon := \mathbb{1}_{|x-y| \leq \delta_\varepsilon}$$

$$|(G - \lambda I) f_\varepsilon| \leq \varepsilon |f|$$

$$\Rightarrow \|(G - \lambda I) f_\varepsilon\|_2 \leq \varepsilon \|f_\varepsilon\|_2$$

$\Rightarrow (G - \lambda I)$ is not invertible.

As $\sigma(G)$ is closed we have "$\supseteq$".

$(\subseteq)$: Let $\lambda \in \sigma(G)$ and suppose $\lambda \notin \overline{\{g(y) : y \in \mathbb{R}\}}$.

Then $G - \lambda I$ has a bounded inverse def. with $(G-\lambda I)^{-1} f = \frac{1}{g(x)-\lambda} f(x)$

(it is bounded as $\exists_{\varepsilon > 0}$ $|g(x) - \lambda| \geq \varepsilon$ so that $\frac{1}{|g(x)-\lambda|} \leq \frac{1}{\varepsilon}$).

As $\sigma(G) = \overline{\{ g(y) : y \in \mathbb{R} \}}$ we conclude that G can be compact only in the trivial case $g \equiv 0$.

C1 $A : H \to H$: find B s.t. $B^n = A$.

A is diagonalizable i.e. there is $\{e_k\}_{k\geq 0}$ OB of H and $\{\lambda_k\}$ assos. eigenvalues,

$$A e_k = \lambda_k e_k .$$

Let $Bx := \sum_{k\geq 1}^{\infty} \lambda_k^{1/n} e_k \langle x, e_k \rangle$

The series converges $\forall_x$ as $S_N := \sum_{k\geq N}^{\infty} \lambda_k^{1/n} e_k \langle x, e_k \rangle$

$$\|S_N - S_M\|_H^2 = \sum_{k=N}^{M} \langle x, e_k \rangle^2 \underbrace{\lambda_k^{1/n}}_{\leq C} \to 0 \text{ as } N, M \to \infty$$

Moreover $\|Bx\|^2 \leq \|\lambda_k\|_\infty \|x\|^2 \Rightarrow$ bdd op.

$$B^2 x = \sum_{j\geq 1}^{\infty} \lambda_j^{1/n} e_j \left\langle \sum_{k\geq 1} \lambda_k^{1/n} e_k \langle x, e_k \rangle , e_j \right\rangle$$

$$= \sum_{j\geq 1}^{\infty} \lambda_j^{2/n} e_j \langle x, e_j \rangle$$

$\square$.