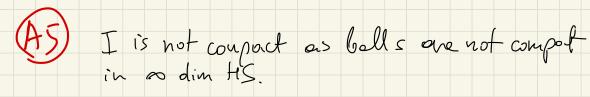
Functional Analysis, PS11 VER: 20.01.2021 (A1) T: E -> F compact => T(B, (0)) is coupact $= \sum_{\substack{||x|| \leq 1}} \sup_{x \in A} ||Tx||_{F} \leq C = \sum_{\substack{||x|| \leq 1}} ||Tx||_{F} \leq C$ ors conpact sets are bounded. (A2) (A) $\overline{T(B_{1}(0))}$ is compared (B) $\{X_{n}, \{6, 6\}\} = \{T_{n}, T_{n}\}$ has conv. subseq. $(A) \Longrightarrow (B): M = \sup_{n} ||x_n||, \text{ then } \frac{x_n}{M} \in B_1(0)$ $\Rightarrow \frac{T(x_m)}{M} \in \overline{T(B_1(0))} \Rightarrow \frac{T(x_m)}{M} \Rightarrow a$ $\Rightarrow T(x_{mk}) \rightarrow \alpha M.$

 $(B) \rightarrow (A):$ let {ym} ~ T(B(0,1)). We want to find on subseq. Ynk convergent in F. Choose $t_n \in T(B(0,1))$, $||z_n - y_n|| \leq \frac{1}{n}$. By (B) z_n has a converging subsequence. 2mk ? Z. But then ynk ? 2 too since $\|y_{n_{k}}-z\| \leq \frac{1}{n_{k}} + \|z-z_{n_{k}}\| \to 0$ as $n_{k} \to \infty$. A) T(E→F S:E→F compact
→ T+S is compact. {xm ? lobd in E. Choose subseq. s.t. {Txme } Convergesond another one s.t. SSKnik, 3 con. It follows that {(T+S) xm/ce 3 is convergent.

(A4) ge ([0]) $T: (To_1] \rightarrow CTo_1]$ $(Tf)(x) = \int_{x}^{x} f(y)g(y) dy$ Let 2 fru 3 nr. IN be bad in (IO1). Ve need to find subseq. { Tfruk ? converging in (IO1). We use A-A: (1) $|f_{f_m}(x)| \leq ||f_m||_{\infty} ||g||_{\infty} \leq ||g||_{\infty} \sup_{n} ||f_n||_{\infty} \leq C.$ $(2)|(f_{f_{n}})(x) - (f_{f_{n}})(y)| = \int_{x}^{y} f_{n}(y)g(y) dy$ $\leq |y-x| \|f_n\|_{\infty} \|g\|_{\infty} \leq C |y-x|.$ D.



(A6)

Otherwise I vould be compact ors

I = (I - T) + T.

Hilbert -Schmidt operators $Tf(x) = \int K(x,y) f(y) dy$ If Kis continuous, if was discussed in the lecture that T is compact. Now, we assume that $K \in L^2(A \times A)$. Note that this means that for a.e. $X \in A$ $\int K^2(x,y) dy < \infty$. (i.e. $K(x, \cdot) \in L^2(A)$). Again, let for be bodd in $L^2(A)$. Choose a subsequence of $\{f_n\}$ converging healthy in $L^2(A)$, by Banach-Alazglu, , i.e. $f_m \rightarrow f$.

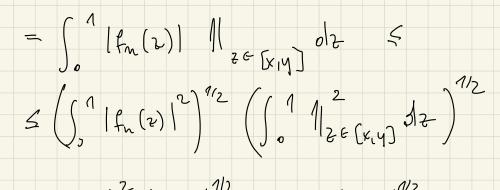
As $K(x_1) \in [2^{\circ}(\mathcal{R}) \text{ for o.e. } x \in \mathcal{R} \text{ j for o.e. } x \in \mathcal{R}$ $\int_{\mathcal{R}} K(x_1y) f_{n_k}(y) dy \implies \int_{\mathcal{R}} K(x_1y) f(y) dy \quad (a.e. x \in \mathcal{R}).$ So we obtained $Tf_{m_k} \longrightarrow Tf$ a.e. $x \in \mathcal{R}$. We need to upgrade this convergence to $L^2(\mathcal{R})$. Note that $|Tf_{m_k}(x)| \leq \int_{\mathcal{R}} |K(x_1y)| |f_{m_k}(y)| dy \leq ||f_{m_k}||_2 ||K(x_1y)||_2 \leq C ||K(x_1)||_2 = C (\int_{\mathcal{R}} K^2(x_1y) dy)^{N_2}$ $\Rightarrow |Tf_{m_k}(x)|^2 \leq C^2 \int_{\mathcal{R}} K^2(x_1y) dy \ll \text{integrable majorant}$ $\Rightarrow Tf_{m_k} \longrightarrow Tf \text{ in } L^2(\mathcal{R}) \text{ by dominated convergence.} \qquad \Box.$

 $\frac{B2}{(Tf)(x)} = \int_0^x f(y) \, dy$

We first prove that T: L2(Q1) -> L2(Q1) is compact. Let Efn? ledd in (?(0,1). Ire need to prove that ST fn? has converging subsequence in L2(0,1) - First, we find this subsequence in (TP1].

 $|| Tf_m ||_{\infty} \leq \int_{0}^{1} |f_{n}(y)| dy \leq \left(\int_{0}^{1} |f_{n}(y)| dy\right)^{1/2}$ $\leq || f_m ||_2.$

 $|tf_{n}(x) - Tf_{n}(y)| = \int |f_{n}(z)|dz =$



 $\leq \|f_{m}\|^{2} |x-y|^{1/2} \leq C |x-y|^{n/2}$

So there is subsequence converging wit 11.11 nom So it converges also wit [1.1], norm. (ne use here that TO, 1] has finite measure). Now, when we know that T is compact, $O \in G(T)$ omolony $\lambda \in G(T)$, $\lambda \neq O$ is its eigenvalue. $Tf = \lambda f$ i.e. $\int_{0}^{x} f(y) dy = \lambda f(x)$ It follows that f is C^{\perp} , f(0)=0 and $\begin{cases} f'(x) = \frac{1}{\lambda} f(x) \\ f(0) = 0 \end{cases}$

There is exactly one f solving this ODE: f=0

Π.

(B4) First, we find 6(G) $\sigma(G) = \{ q(x) : x \in IR \}$ 2: If $\lambda = g(y)$ Fyer then $(G - \Lambda D)f$ $= (g(x) - \lambda)f(x) = (g(x) - g(y))f(x)$ g is continuous so It J. J. V. Ix-y 135 $= \left| \left| g(x) - g(y) \right| \leq \varepsilon$. y is fixed. $f_{\varepsilon} := 1 |_{|x-y| \le \delta_{\varepsilon}}$ $|(G-\Lambda I)f_{g}| \leq \mathcal{E}|f|$ $\Rightarrow ||(G - \Lambda I)f_{\varepsilon}||_{2} \leq \varepsilon ||f_{\varepsilon}||_{2}$ => (G-XI) is not invertible. Hs G(G) is closed we have "2".

(E) Let A = 6 (G) and suppose A & Eg(y): yell. Then G-AI has a bounded inverse def. with $(G - \lambda T)f = \frac{1}{g(x) - \lambda} f(x)$ (it is bounded as $f_{\epsilon>0}$ (g(x)- λ) $\geq \epsilon$ so that $\frac{1}{|g(x)-\lambda|} \leq \frac{1}{\xi}$. As $G(G) = \{g(y): y \in \mathbb{R}\}\$ we conclude that G can be compact only in the trivial case g = 0.

(C1) A: $H \rightarrow H$: find B s.t. B'' = A. A is diagonalizable i.e. there is $\{e_k\}_{k\geq 0}$ OB of H and $\{\lambda_k\}$ assos. aigenvalues, Aek = Kkek. Let $B_{x'} = \sum_{k \ge 1}^{\infty} \lambda_{k}^{1/n} e_{k} < v_{1}e_{k} >$ The series converges \forall_X vis $S_N := \sum_{k \ge N} \frac{1}{k} e_k \langle x, e_k \rangle$ $\|S_N - S_M\|_{H}^2 = \sum_{k=N}^{M} \langle x_i e_k \rangle^2 \frac{1}{\lambda_k} \longrightarrow O \text{ as } N, M \longrightarrow \infty$ Noveover $||B_{x}||^{2} \leq ||\lambda|| \leq C^{2} = bold op.$ $B_{\chi}^{2} = \sum_{i=1}^{\infty} \lambda_{ij}^{1/m} e_{ij} \leq \sum_{k \geq 1}^{1/m} e_{ik} \langle x_{1}, e_{ik} \rangle, e_{ij} \rangle$ $= \sum_{j\geq l}^{\infty} \lambda_j^{2/n} e_j < \times_l e_j >$ Ħ