Functional Analysis, PS12

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A1 This is just revision. For all details, see

- P. Strzelecki "Mathematical Analysis II"
- L. C. Evans, M. Gariepy "Measure theory and fine properties of functions" Section 4.1.

this is somehow average of f with g determing weights

Let $f \in L^1$, $g \in C_0^k(\mathbb{R}^n)$. Note that $f * g(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy$

Roughly speaking, as x is only in g, differantiation does not see f. More precisely

$$f*g(x+h) - f*g(x) = \int_{\mathbb{R}^n} f(y) \underbrace{\left[g(x+h-y) - g(x-y) \right]}_{\leq \|Dg\|_\infty \cdot |h|} dy$$

So by Dominated Convergence Theorem ($f \in L^1$)

$$f*g(x+h) - f*g(x) - f*(Dg\cdot h)(x) \longrightarrow 0$$

So $D(f*g) = f * Dg$. Similarly higher derivatives ...

A2 $\qquad \|f*g\|_r \le \|f\|_p \|g\|_q \qquad 1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$

Fix x.

$$|f*g| \le \int |f(x-y)|\,|g(y)|\,dy \le$$

$$\le \int |f(x-y)|^{p/r}\,|g(y)|^{q/r}\,|f(x-y)|^{1-p/r}\,|g(y)|^{1-q/r}$$

$$\underset{\text{Hölder}}{\le} \underbrace{\left\|\left(f(x-y)^p\, g(y)^q\right)^{1/r}\right\|_r}_{\mathrm{I}} \; \overbrace{\left\|f(x-y)^{1-\frac{p}{r}}\right\|_{\frac{rp}{r-p}}}^{\mathrm{II}}$$

$\frac{r-p}{r}$

$$\left\|g(y)^{1-q/r}\right\|_{\frac{r-q}{qr}}$$

This is OK as $\quad 1 = \frac{1}{r} + \frac{r-p}{rp} + \frac{r-q}{rq} =$

$$= \frac{1}{r} + \frac{1}{p} - \frac{1}{r} + \frac{1}{q} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1.$$

$$\mathrm{I} \le \left(\int |f(x-y)|^p |g(y)|^q\right)^{1/r}$$

$$\mathrm{II} \le \left(\int |f(x-y)|^p\right)^{\frac{r-p}{rp}} \le \|f\|_p^{\frac{r-p}{r}}$$

$$\mathrm{III} \le \left(\int |g(y)|^q\right)^{\frac{r-q}{qr}} \le \|g\|_q^{\frac{r-q}{r}}$$

$$\|f*g\|_r^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \iint |f(x-y)|^p |g(y)|^q \, dx\, dy$$

$$\le \|f\|_p^{r-p} \|g\|_q^{r-q} \|g\|_q^q \|f\|_p^p$$

$$= \|f\|_p^r \|g\|_q^r$$

This is generalization of inequality from lecture

$(f \in L^p,\ g \in L^1 \Rightarrow f*g \in L^p)$

A3 $f \in L^1,\ g \in L^\infty,\ g$ is Lipschitz

$$\|f*g\|_\infty \le \|f\|_1 \|g\|_\infty < \infty.$$

$$|(f*g)(x) - (f*g)(y)| =$$

$$= \left| \int f(z) g(x-z)\, dz - \int f(z) g(y-z)\, dz \right|$$

$$\le \int |f(z)|\, |x-y|\, dz \le \|f\|_1 |x-y| \quad \checkmark.$$

A4

$$\int_{\mathbb{R}^d} f(x)\; g * h(x) =$$

$$= \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} g(x-y)\, h(y)\, dy\, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)\, g(y-x)\, dx\;\; h(y)\, dy = \int f * g(y) h(y)\, dy$$

B1 $f * \eta_\varepsilon \rightrightarrows f$ on compact subsets, f continuous

$$f(x) - f * \eta_\varepsilon(x) = f(x) - \int f(x-y)\, \eta_\varepsilon(y)\, dy$$

$$= \int \big(f(x) - f(x-y) \big)\, \eta_\varepsilon(y)\, dy$$

Note that $|y| \leq \varepsilon$. Hence in compact set f is unif. cont. and the cond. is clear.

B3 $f \in L^1(\Omega)$. $\int_\Omega f\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega) \Rightarrow$

$\Rightarrow f = 0$ in Ω.

$$\varphi_\varepsilon = \left[(\operatorname{sgn} f)\, \mathbb{1}_{\bar\Omega}\right] * \eta_\varepsilon \in C_c^\infty(\Omega)$$

$$\Rightarrow \quad \int_\Omega f\varphi_\varepsilon = 0$$

$$\downarrow$$

$$\int_\Omega f \operatorname{sgn} f\, \mathbb{1}_{\bar\Omega} = 0 \quad \Rightarrow \quad f = 0.$$

B2 $f \in L^p(\mathbb{R})$

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

$$\left(1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\right).$$

$$(f * \eta_\varepsilon)^{(k)} = f * \eta_\varepsilon^{(k)}$$

$$\|(f * \eta_\varepsilon)^k\|_p \leq \|f\|_p \|\eta_\varepsilon^{(k)}\|_1$$

$$\eta_\varepsilon^{(1)} = \left(\frac{1}{\varepsilon} \eta\left(\frac{x}{\varepsilon}\right)\right)' = \frac{1}{\varepsilon^2} \eta'\left(\frac{x}{\varepsilon}\right)$$

$$\eta_\varepsilon^{(k)} = \frac{1}{\varepsilon^{k+1}} \eta^{(k)}\left(\frac{x}{\varepsilon}\right)$$

$$\|\eta_\varepsilon^{(k)}\|_1 = \frac{1}{\varepsilon^{k+1}} \int_{\mathbb{R}} \eta^{(k)}\left(\frac{x}{\varepsilon}\right) dx \underset{y=\frac{x}{\varepsilon}}{=}$$

$$= \frac{1}{\varepsilon^k} \int \eta^{(k)}(y)\, dy = C_k \cdot \frac{1}{\varepsilon^k}.$$

$$\Rightarrow \quad \|(f * \eta_\varepsilon)^{(k)}\|_p \leq \|f\|_p \cdot \frac{C_k}{\varepsilon^k}.$$

$$\|(f * \eta_\varepsilon)^{(k)}\|_\infty \leq \|f\|_p \|\eta_\varepsilon^{(k)}\|_{p'}$$

$$\|\eta_\varepsilon^{(k)}\|_{p'} = \frac{1}{\varepsilon^{k+1}} \left[\int_{\mathbb{R}} \left(\eta^{(k)}\left(\frac{x}{\varepsilon}\right)\right)^{p'} dx\right]^{1/p'} =$$

$$= \frac{1}{\varepsilon^{k+1}} \left[\int \eta^{(k)}(y)^{p'}\right] \varepsilon^{1/p'} \leq \ldots \quad \checkmark.$$

S3 $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow f \in L^p(\mathbb{R}^d) \quad \forall_{1 \leq p \leq \infty}$

$p = \infty$ done as f is bounded.

$$\int_{\mathbb{R}^d} |f|^p \leq \underbrace{\int_{B(0,1)} |f|^p}_{\leq \|f\|_\infty^p \, |B(0,1)|} + \int_{\mathbb{R}^d \setminus B(0,1)} |f|^p$$

on this part we use that $|f|\,|x|^\alpha \leq C_\alpha$ for any α.

$$\int_{\mathbb{R}^d \setminus B(0,1)} |f|^p \leq \int_{\mathbb{R}^d \setminus B(0,1)} \frac{C_\alpha^p}{|x|^{p\alpha}} = C_\alpha^p \int_1^\infty \frac{1}{r^{p\alpha}} C \cdot r^{d-1} \, dr = \tilde{C} \int_1^\infty r^{d-1-p\alpha} \, dr$$

We want $d-1-p\alpha < -1 \Rightarrow \alpha > \frac{d}{p}$ will work to make this integral finite.