Functional Analysis, PS13

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Literature:
- basics: J. Duoandikoetxea "Fourier Analysis" chap. 1
- more adv. topics Grafakos "Classical Fourier Analysis" chap. 2.2–2.4

1 Clearly, Fourier transform is linear.

A) $$\left| \int_{\mathbb{R}^n} f(x) \underbrace{e^{-2\pi i \xi \cdot x}}_{|\dots| \leq 1} dx \right| \leq \|f\|_1$$

B) Let $\xi_n \to \xi$ in $\mathbb{R}^n$. We want $\hat{f}(\xi_n) \to \hat{f}(\xi)$. Indeed,

$$\hat{f}(\xi_n) = \int_{\mathbb{R}^n} \underbrace{f(x) e^{-2\pi i \xi_n \cdot x}}_{\text{converges pointwise to } f(x) e^{-2\pi i \xi x}} dx \longrightarrow \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = f(\xi)$$

; integrable majorant is f by DCT.

C) $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ $\quad (f \in L^1(\mathbb{R}^n))$.

$$\overline{C_c^\infty(\mathbb{R}^n)}^{L^1} = L^1(\mathbb{R}^n) \quad (\text{i.e. } C_c^\infty(\mathbb{R}^n) \text{ is dense in } L^1(\mathbb{R}^n)).$$

Let $\xi \in \mathbb{R}^n$, $|\xi| \to \infty$. In particular, $\exists_{i \in \{1,\dots,n\}}$ $|\xi_i| \to \infty$.

Then, if $f \in C_c^\infty(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n} f(x) \frac{1}{(-2\pi i \xi_i)} \partial_{x_i}\left[e^{-2\pi i \xi x} \right] dx$$

$$\underset{\substack{\uparrow \\ \text{integration by} \\ \text{parts}}}{=} -\frac{1}{2\pi i \xi_i} \int_{\mathbb{R}^n} \partial_{x_i} f(x)\, e^{-2\pi i \xi x} dx \leq \frac{1}{2\pi |\xi_i|} \underbrace{\| \partial_{x_i} f \|_{L^1}}_{\text{finite as } f \in C_c^\infty(\mathbb{R}^n)} \longrightarrow 0$$

The general statement follows by density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ (this time check it yourself!).

Ok: Let $f_n \in C_c^\infty(\mathbb{R}^n)$, $f_n \to f$ in $L^1(\mathbb{R}^n)$, $\| \hat{f}_n - \hat{f} \|_1 \to 0, \dots$.

2)

A) $f * g(\cancel{x}) \in L^1(\mathbb{R}^n)$ if $f, g \in L^1(\mathbb{R}^n)$ [Young's inequality]

$$\widehat{f*g}(\xi) = \int_{\mathbb{R}^n} f*g(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) g(x-y) e^{-2\pi i \xi x} dy\, dx$$

$$= \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} g(x-y) e^{-2\pi i \xi (x-y)} dx \right] e^{-2\pi i \xi y} dy =$$

$$= \left[\int_{\mathbb{R}^n} f(y) e^{-2\pi i \xi y} dy \right] \hat{g}(\xi) = \hat{f}(\xi) \hat{g}(\xi).$$

So $\boxed{f, g \in L^1(\mathbb{R}^n)} \Rightarrow f*g \in L^1(\mathbb{R}^n)$ and $\widehat{f*g}(\xi) = \hat{f}(\xi) \hat{g}(\xi)$.

B) $\tau_h f(x) = f(x+h)$ $\boxed{f \in L^1(\mathbb{R}^n)}$

$$\widehat{\tau_h f}(\xi) = \int_{\mathbb{R}^n} \tau_h f(x) e^{-2\pi i \xi x} dx = \int_{\mathbb{R}^n} f(x+h) e^{-2\pi i \xi x} dx =$$

$$= \left[\int_{\mathbb{R}^n} f(x+h) e^{-2\pi i \xi (x+h)} dx\right] e^{2\pi i \xi h} = \hat{f}(\xi) e^{2\pi i \xi h}.$$

C) $f \in L^1(\mathbb{R}^n)$, $\partial_{x_j} f \in L^1(\mathbb{R}^n)$, f vanishes at ∞ suff fast ($f \in S(\mathbb{R}^n)$ works)

$$\widehat{\partial_{x_j} f}(\xi) = \int_{\mathbb{R}^n} \partial_{x_j} f(x) e^{-2\pi i \xi x} = -\int_{\mathbb{R}^n} f(x)(-2\pi i \xi_j) e^{-2\pi i \xi x} dx =$$

$$= 2\pi i \xi_j \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx = 2\pi i \xi_j \hat{f}(\xi)$$

buttons.

for this, see Grafakos "Classical Fourier Anal." - Chapt. 2.2-2.4

D) $\delta_h f = f(x/h)$ $f \in L^1(\mathbb{R}^n)$

$$\widehat{\delta_h f}(\xi) = \int_{\mathbb{R}^n} \delta_h f(x) e^{-2\pi i \xi x} dx =$$

$$= \int_{\mathbb{R}^n} f(x/h) e^{-2\pi i \xi x} dx \underset{y = \frac{x}{h}}{=} \left[\int_{\mathbb{R}^n} f(y) e^{-2\pi i (\xi h)\cdot y} dy\right] h^n$$

$$= h^n \hat{f}(\xi h).$$

3A) We first note that one-dimensional case is sufficient. Indeed, $f = e^{-\pi|x|^2}$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\pi(x_1^2 + \dots + x_n^2)} e^{-2\pi i(\xi_1 x_1 + \dots + \xi_n x_n)} =$$

$$= \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\pi x_i^2} e^{-2\pi i \xi_i x_i} \, dx_i$$

So we work on $\mathbb{R}$ instead of $\mathbb{R}^n$, $f(x) = e^{-\pi x^2}$, $f : \mathbb{R} \to \mathbb{R}$.

f satisfies $\begin{cases} f' + 2\pi x f(x) = 0 \\ f(0) = 1. \end{cases}$ $(*)$

Moreover, $\hat{f}(0) = \int_{\mathbb{R}^n} e^{-\pi x^2} = 1$. We want to prove that $\hat{f}$ also solves $(*)$. Indeed,

- $\frac{d}{d\xi} \hat{f}(\xi) = \int_{\mathbb{R}} f(x)(-2\pi i x) e^{-2\pi i \xi x} \, dx = \widehat{f(-2\pi i x)}(\xi)$.
- $2\pi \xi \hat{f}(\xi) = +\frac{1}{i}(2\pi \xi i)\hat{f}(\xi) = +\frac{1}{i}\widehat{f_x}(\xi) = -i\widehat{f_x}(\xi)$.

Therefore $\frac{d}{d\xi}\hat{f}(\xi) + 2\pi\xi\hat{f}(\xi) = \left(-2\pi i x f(x) - i f_x\right)^\wedge(\xi) = 0$

By uniqueness of slns to $(*)$, the assertion follows. $\blacksquare$

3B) $f(x) = e^{-x} \mathbb{1}_{x>0}$ $\quad f: \mathbb{R} \to \mathbb{R}$

$$\hat{f}(\xi) = \int_0^\infty e^{-x} e^{-2\pi i \xi x} \, dx = \int_0^\infty e^{-x(2\pi i \xi + 1)} \, dx =$$

$$= \frac{1}{1 + 2\pi i \xi} \notin L^1(\mathbb{R})$$

So it is not true that $f \in L^1(\mathbb{R}^n) \Rightarrow \hat{f} \in L^1(\mathbb{R}^n)$.

4) Let $f \in S(\mathbb{R}^n)$. We find $u \in S(\mathbb{R}^n)$ s.t.

$$-\Delta u + u = f$$

First, take Fourier transform to get $\hat{f}(\xi) = 4\pi^2 |\xi|^2 \hat{u}(\xi) + \hat{u}(\xi)$

$$\Rightarrow \hat{u}(\xi) = \frac{1}{4\pi^2|\xi|^2 + 1} \hat{f}(\xi).$$

As $f \in S(\mathbb{R}^n) \Rightarrow \hat{f}(\xi) \in S(\mathbb{R}^n) \Rightarrow \dfrac{\hat{f}(\xi)}{1 + 4\pi^2|\xi|^2} \in S(\mathbb{R}^n)$.

Since Fourier transform is isomorphism on $S(\mathbb{R}^n)$, there is $u \in S(\mathbb{R}^n)$ s.t. $\hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{1 + 4\pi^2|\xi|^2}$. We write

$$u(x) = \left(\frac{\hat{f}(\xi)}{1 + 4\pi^2|\xi|^2} \right)^{\vee}.$$

5)

$$f = u + \partial_1^2 \partial_2^2 \partial_3^4 u + 4i\, \partial_1^2 u + \partial_7^2 u$$

$$\widehat{(\partial_1^2 \partial_2^2 \partial_3^4 u)}(\xi) = (2\pi i)^8 \xi_1^2 \xi_2^2 \xi_3^4 \hat{u}(\xi)$$

$$= \overset{c_1}{(2\pi)^8} \xi_1^2 \xi_2^2 \xi_3^4 \hat{u}(\xi)$$

$$\widehat{(4i\partial_1^2 u)}(\xi) = 4i\,(2\pi i)^2 \xi_1^2 \hat{u}(\xi)$$

$$= \overbrace{4(2\pi)^2}^{c_2} (-1)\, i\, \xi_1^2 \hat{u}(\xi)$$

$$\widehat{(\partial_7^2 u)}(\xi) = (2\pi i)^7 \xi_2^2 \hat{u}(\xi) =$$

$$= \underbrace{(2\pi)^7}_{c_3} (-i)\, \xi_2^2 \hat{u}(\xi)$$

$$\hat{f} = \hat{u} + c_1 \xi_1^2 \xi_2^2 \xi_3^4 \hat{u} - i\, c_2 \xi_1^2 \hat{u}$$

$$- i\, c_3 \xi_2^2 \hat{u}$$

$$\hat{u} = \hat{f} \Big/ 1 + c\, \xi_1^2 \xi_2^2 \xi_3^4 + i\left(-c_2 \xi_1^2 + c_3 \xi_2^2\right)$$

Note that this polynomial is bdd from below.

6) $1 = \int 1\cdot|\psi|^2 = -\int x \frac{d}{dx}|\psi|^2 = -\int x \frac{d}{dx}\psi\bar{\psi}\,dx$

$= -\int x\,\psi_x\bar{\psi} - \int x\,\psi\,\bar{\psi}_x \leq 2\left(\int|x\,\psi(x)|^2\right)^{1/2}\left(\int|\psi_x|^2\right)^{1/2}$

By Plancherel $\int|\psi_x|^2 = \int|\widehat{\psi_x}|^2 = 4\pi^2\int|\hat{\psi}(\xi)\,\xi|^2$

so that $1 \leq 2\cdot(2\pi)\left(\int|x\,\psi(x)|^2\right)^{1/2}\left(\int|\xi\,\hat{\psi}(\xi)|^2\right)^{1/2}$

$\Rightarrow \frac{1}{16\pi^2} \leq \left(\int|x\,\psi(x)|^2\right)^{1/2}\left(\int|\xi\,\hat{\psi}(\xi)|^2\right)^{1/2}$ $\square$.