Functional Analysis, PS 1

VER: 5. 11. 2020

A1

- $C[0,1]$ – yes
- $C(0,1)$ – no, $\frac{1}{x} \in C(0,1)$ but $\|\frac{1}{x}\|_\infty = \infty$.
  so $\|\cdot\|_\infty$ is not a norm on $C(0,1)$.
- $C(\mathbb{R})$ – no, $x \in C(\mathbb{R})$ but $\|x\|_\infty = \infty$.

A2 No, $x \in C[0,1]$, $x \neq 0$ but $|x|_{C^1} = 0$.

A3 (a) $\forall_{\varepsilon > 0} \; \exists_{N_\varepsilon} \; \forall_{n,m \geq N_\varepsilon} \; \|x_n - x_m\| \leq \varepsilon$.

Use this with $\varepsilon = 1$. If $k \geq N_\varepsilon$ we have

$$\|x_k\| \leq \|x_{N_1}\| + \|x_{N_1} - x_k\| \leq 1 + \|x_{N_1}\| < \infty$$

Otherwise, $\|x_k\| \leq \sup_{1 \leq i \leq N_\varepsilon} \|x_i\| < \infty$ which is finite because supremum is taken over finitely many terms.

(b) [redacted]: If $\{x_n\}$ is Cauchy and it has convergent subsequence then the whole sequence converges. Indeed, let $\varepsilon > 0$, $x_{n_k} \to x$ be convergent subsequence. There is N s.t. $\forall_{n,m \geq N}$ $\|x_n - x_m\| \leq \frac{\varepsilon}{2}$. There is also M s.t. $\forall_{n_k \geq M}$ $\|x_{n_k} - x\| \leq \varepsilon/2$.

[redacted]. Take any $n_k \geq \max(N, \varepsilon)$. Then,

$$\|x - x_n\| \leq \|x - x_{n_k}\| + \|x_{n_k} - x_n\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall_{n \geq N}.$$

✓

A4

[redacted] (exercise in Topology) $Y \subset X$, $(X, \|\cdot\|_X)$ Banach space

$(Y, \|\cdot\|_X)$ is Banach $\iff$ Y is closed in $(X, \|\cdot\|_X)$.

($\Leftarrow$) Let $(y_n) \subset Y$ be Cauchy sequence. Then, since $Y \subset X$, it is also Cauchy in X and since it is Banach space, it has limit in X. Call it $x \in X$. By closedness of Y, $x \in Y$ and assertion follows,

($\Rightarrow$) Let $(y_n) \subset Y$ be converging sequence, i.e. $y_n \to x$ in $\|\cdot\|_X$ for some $x \in X$. Since any converging sequence is Cauchy, $x \in Y$. □

This fact is quite simple but allows to handle many Banach spaces.

A5

[redacted]

X Banach space $\iff \left\{ \sum_{k=1}^{\infty} \|x_k\| < \infty \Rightarrow \sum_{k=1}^{\infty} x_k \text{ converges in } X \right\}$

($\Leftarrow$) Let (x_k) be a Cauchy sequence in X. [redacted] We only need to check that it has convergent subsequence ([redacted])

Choose subsequence s.t. $\|x_{k_{n+1}} - x_{k_n}\| \leq 2^{-k}$. $\Rightarrow \sum \|x_{k_{n+1}} - x_{k_n}\| < \infty$

$\Rightarrow \sum x_{k_{n+1}} - x_{k_n}$ converges in X $\Rightarrow \{x_{k_n}\}$ converges in X $\Rightarrow$ $\{x_n\}$ converges in X. (as this is Cauchy sequence).

($\Rightarrow$) it is easy as $\left(\sum_{k=1}^{n} x_k\right)_{n \in \mathbb{N}}$ is a Cauchy sequence .... □.

Recall $L^p(X, \mathcal{F}, \mu)$, $L^p(\Omega)$, $L^p(a,b)$.

Recall why $L^p(a,b) = L^p[a,b]$.

[redacted] (standard application of Hölder) Let $p \geq q$. Then

B1 $\|f\|_q^q = \int |f|^q = \int |f|^q \cdot 1 \leq \left(\int |f|^p\right)^{q/p} \left(\int 1\right)^{p-q/p}$

Hölder with $\frac{p}{q} \leadsto 1 = \frac{1}{\left(\frac{p}{q}\right)} + \frac{1}{?} \Rightarrow \frac{1}{?} = \frac{p-q}{p}$

$\Rightarrow \|f\|_q \leq \|f\|_p \cdot \left[\mu(X)\right]^{p-q/pq}$

Note that $\frac{1}{x} \in L^2(1,\infty)$ but $\frac{1}{x} \notin L^1(1,\infty)$.

[redacted] (standard app. of L4)

B2 Observe that by L3, set $L^2(0,1)$ is subset of $L^1(0,1)$. Therefore,

if $(L^2(0,1), \|\cdot\|_1)$ were Banach space, $L^2(0,1)$ would be closed in $(L^1(0,1), \|\cdot\|_1)$.

The point is that there are fcns f in L^1 but not in L^2, for instance $\frac{1}{\sqrt{x}} \in L^1(0,1)$ but $\frac{1}{\sqrt{x}} \notin L^2(0,1)$. To be more precise, let

$$f_n(x) = \min\left(\frac{1}{\sqrt{x}}, n\right)$$

Claim: $f_n \to \frac{1}{\sqrt{x}}$ in L^1. $\Leftrightarrow \int \left| f_n - \frac{1}{\sqrt{x}} \right| \to 0$ by dominated convergence. This contradicts closedness of L^2 in $(L^1, \|\cdot\|_1)$.

In general case, we consider $\frac{1}{x^{1/p}}$ instead of $\frac{1}{x^{1/2}}$.

B3 $\leadsto$ HOMEWORK

B4 $1 \le p_0 < p \le p_1 < \infty$, $f \in L^p$,

$$f_0 = f \mathbb{1}_{|f| \ge 1} \qquad f_1 = f \mathbb{1}_{|f| \le 1}.$$

$1 \le \frac{1}{|f|}$

$f_1 \in L^{p_1}$ since $\displaystyle\int |f_1|^{p_1} = \int |f|^{p_1} \mathbb{1}_{|f| \le 1} \le \int |f|^{p_1} \left(\frac{1}{|f|}\right)^{p_1 - p}$

$$= \int |f|^p < \infty$$

$1 < |f|$

$f_0 \in L^{p_0}$ since $\displaystyle\int |f_0|^{p_0} = \int |f|^{p_0} \mathbb{1}_{|f| > 1} \le \int |f|^{p_0} |f|^{p - p_0}$

$$\le \int |f|^p < \infty.$$

C1 Let f_n be Cauchy sequence in $C^1[0,1]$. Then f_n, f_n' are Cauchy in $C[0,1]$ so by completeness, there are f, g s.t. $f_n \to f$, $f_n' \to g$ uniformly. We have to check that $g = f'$.
Indeed, for $n \in \mathbb{N}$, $f_n(t) = f_n(0) + \int_0^t f_n'(s)\,ds$ and using uniform convergence, $f(t) = f(0) + \int_0^t g(s)\,ds \Rightarrow f' = g$ as desired. $\square$

C2 Suppose that $(P, \|\cdot\|_\infty)$ is a Banach space. Note that $\sum \frac{x^k}{k!}$ converges absolutely $\left(\sum \frac{\|x^k\|}{k!} \le \sum \frac{1}{k!} = e^1\right)$. By AS $\sum \frac{x^k}{k!}$ converges in P wrt $\|\cdot\|_\infty$. But we know $\sum \frac{x^k}{k!} = e^x$ and $e^x \notin P$.

Detailed solution to C1

It can be checked that this is normed space. To prove that it is a Banach space, we need to demonstrate it is complete.

Let $\{f_n\}$ be a Cauchy sequence in $\left(C^1[0,1], \|f\|_\infty + \|f'\|_\infty\right)$.
We need to prove that it converges in this space, i.e. there is $f \in C^1[0,1]$ such that

(*) $\|f_n - f\|_\infty + \|f_n' - f'\|_\infty \to 0$ when $n \to \infty$.

First, we note that

$$\forall_{\varepsilon>0} \ \exists_{N(\varepsilon)} \ \forall_{n,m \geq N(\varepsilon)} \ \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty \leq \varepsilon$$

In particular,

$$\forall_{\varepsilon>0} \ \exists_{N(\varepsilon)} \ \forall_{n,m \geq N(\varepsilon)} \ \|f_n - f_m\|_\infty \leq \varepsilon, \ \|f_n' - f_m'\|_\infty \leq \varepsilon.$$

Therefore, $\{f_n\}$, $\{f_n'\}$ are Cauchy sequences in $(C[0,1], \|\cdot\|_\infty)$. As this space is complete, there are $g, h \in C[0,1]$ such that

(*) • $f_n \to g$ in $C[0,1]$, i.e. $\|f_n - g\|_\infty \to 0$ as $n \to \infty$

(*) • $f_n' \to h$ in $C[0,1]$, i.e. $\|f_n' - h\|_\infty \to 0$ as $n \to \infty$.

If we knew that $g \in C^1[0,1]$ and $g' = h$, we could deduce (*).

To see this, we note that

$$f_n(t) = f_n(0) + \int_0^t f_n'(s)\,ds$$

By convergence (*), we have $f_n(t) \to g(t)$, $f_n(0) \to g(0)$.
Moreover, by (*), we also have $\int_0^t f_n'(s)\,ds \to \int_0^t h(s)\,ds$.
(Indeed: $\left| \int_0^t f_n'(s)\,ds - \int_0^t h(s)\,ds \right| \leq \int_0^t |f_n'(s) - h(s)|\,ds \leq$

$\leq 1 \cdot \| f_n' - h \|_\infty \to 0$).

We conclude $g(t) = g(0) + \int_0^t h(s)\,ds$ which implies $g \in C^1[0,1]$ and $g' = h$.

$\square$.

Remark to C1

(remark 16.10.2020).

The following result is discussed in Analysis I:

THEOREM Let $\{f_n\} \subset C^1[0,1]$ such that

- $f_n(0)$ converges to some $a \in \mathbb{R}$.
- f_n' converges uniformly to some $g \in C[0,1]$.

Then there is $f \in C^1[0,1]$ such that $f_n \to f$ uniformly and $f' = g$.

PROOF: $f_n(t) = f_n(0) + \int_0^t f_n'(s)\,ds$

$$\Rightarrow \forall_{t \in [0,1]} \lim_{n\to\infty} f_n(t) = a + \int_0^t g(s)\,ds.$$

Let $f(t) := a + \int_0^t g(s)\,ds$. This function is $C^1[0,1]$.

Moreover

$$\|f_n - f\|_\infty \le |f_n(0) - a| + 1 \cdot \|f_n' - g\| \to 0$$

as $n \to \infty$

■.

Remark: The proof presented in Analysis I is much more complicated as it is performed without the notion of integral.

C3

Then, $C^1[0,1]$ is closed in $(C[0,1], \|\cdot\|_\infty)$.

• $\|f\|_A$ is a norm on $C^1[0,1]$ as it is a norm on the bigger space $C[0,1]$. Suppose $(C^1[0,1], \|\cdot\|_A)$ is Banach. Let f be

(i.e. $|x-\frac{1}{2}|\cdot 2$). Let p_n be a seq of polynomials s.t. $\|f-p_n\|_\infty^2 \to 0$. Note that $p_n \in C^1([0,1])$ so we get contradiction.

C4 ⇝ HOMEWORK

C5

First, $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a normed space: the proof is similar as for $(C[0,1], \|\cdot\|_\infty)$, we only need to check that $\|f\|_\infty < \infty$ for $f \in C_0(\mathbb{R})$. But this follows from vanishing at infinity. Indeed, there is R such that $|f(x)| \leq 1$ for $|x| > R$.

Then
$$\|f\|_\infty \leq \underbrace{\sup_{|x|\leq R} |f(x)|}_{<\infty \text{ by continuity}} + \underbrace{\sup_{|x|>R} |f(x)|}_{\leq 1} < \infty.$$

To prove that $(C_0(\mathbb{R}), \|\cdot\|_\infty)$ we need to establish completeness. Let $\{f_n\}$ be a Cauchy sequence.

CANDIDATE FOR THE LIMIT: We use completeness of $C[-R,R]$ for each $R>0$. We have that $\{f_n\}$ is also Cauchy in $(C[-R,R], \|\cdot\|_\infty)$ so it converges in $C[-R,R]$ to some $f^R \in C[-R,R]$ $\forall_{R\in\mathbb{N}}$. By uniqueness of the limit $f^{R+1}(x) = f^R(x)$ $\forall_{x\in[-R,R]}$.

Hence, we can define $f(x) = f^R(x)$ $\forall_{x\in[-R,R]}$ and $f\in C(\mathbb{R})$. To conclude, we need to prove that $f\in C_0(\mathbb{R})$ and $\|f_n - f\|_\infty \to 0$.

To see $\|f_n - f\|_\infty \to 0$, we write

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)|$$

I had to switch to liminf as I don't know if lim exists.

$$= \liminf_{m\to\infty} |f_n(x) - f_m(x)| \leq \liminf_{m\to\infty} \|f_n - f_m\|_\infty$$

Hence, $\|f_n - f\|_\infty \leq \liminf_{m\to\infty} \|f_n - f_m\|_\infty$.

Let $\varepsilon>0$. Since $\{f_n\}_n$ is Cauchy, there is $N(\varepsilon)$ such that $\forall_{n,m\geq N(\varepsilon)} \|f_n - f_m\| \leq \varepsilon$. If $n\geq N$, $\liminf_{m\to\infty} \|f_n - f_m\|_\infty \leq \varepsilon$.

It follows that $\|f_n - f\|_\infty \leq \varepsilon \quad \forall\, n \geq N(\varepsilon)$ so that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

To see that $f(x) \to 0$ as $x \to \infty$ we write

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \|f - f_n\|_\infty + |f_n(x)|.$$

Fix $\varepsilon > 0$. Choose n so that $\|f - f_n\|_\infty \leq \frac{\varepsilon}{2}$. Since $f_n \in C_0(\mathbb{R})$, choose $R > 0$ so that $|f_n(x)| \leq \frac{\varepsilon}{2}$ for $|x| > R$. Hence, for such R $|f(x)| \leq \varepsilon$, whenever $|x| > R$.

[redacted]

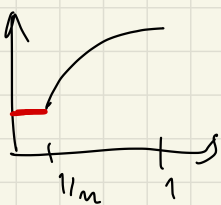
(C6) (A) $(C_{Lip}[0,1], |\cdot|_{Lip})$ is not a Banach space as $1 \in C_{Lip}[0,1]$ but $|1|_{Lip} = 0$.

(B) $(C_{Lip}[0,1], \|\cdot\|_\infty)$ is not a Banach space because norm does not contain information on Lipschitz constant.

→ note that $C_{Lip}[0,1] \subset C[0,1]$

Suppose it is. Let $f(x) = \sqrt{x}$. Define f_n as:

$$f_n(x) = \begin{cases} \sqrt{x} & |x| \geq \frac{1}{n} \\ \sqrt{\frac{1}{n}} & |x| < \frac{1}{n} \end{cases}$$



f_n is Lipschitz continuous on $[0,1]$:

- if $x,y \in [0, \frac{1}{n}]$ $\quad |f_n(x) - f_n(y)| = 0 \leq |x-y|$.
- if $x,y \in [\frac{1}{n}, 1]$ $\quad |f_n(x) - f_n(y)| \leq \| f_n' \|_\infty |x-y|$

  by mean value theorem. The supremum is computed on $[\frac{1}{n}, 1]$. $\quad f_n'(x) = \frac{-1}{2\sqrt{x}} \Rightarrow \|f_n'\|_\infty = \frac{\sqrt{n}}{2}$.
- if $x \in [0, \frac{1}{n}]$, $y \in [\frac{1}{n}, 1]$

$$|f(x) - f(y)| \leq \underbrace{|f(x) - f(\tfrac{1}{n})|}_{=0} + |f(\tfrac{1}{n}) - f(y)| \leq$$

$$\leq \frac{\sqrt{n}}{2} |y - \tfrac{1}{n}| \leq \frac{\sqrt{n}}{2} |y-x| \quad \checkmark.$$

Moreover, $f_n \to f$ wrt $\|\cdot\|_\infty$. Indeed,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,\frac{1}{n}]} \left| \frac{1}{\sqrt{n}} - \sqrt{x} \right| \leq \frac{2}{\sqrt{n}} \to 0$$

as $n \to \infty$.

It follows that $f \in C_{LIP}[0,1]$ which is contradiction!

$$\left.\frac{|f(x) - f(y)|}{|x-y|}\right|_{x=0} = \frac{\sqrt{y}}{y} \to \infty \text{ as } y \to 0 \text{ so}$$

$$|f|_{LIP} = \infty.$$

$\square$.

(c) Let $\{f_n\}$ be Cauchy in $(C_{LIP}[0,1], \|\cdot\|_\infty + |\cdot|_{LIP})$.

$$\forall_{\varepsilon>0} \exists_N \forall_{n,m\geq N} \quad \|f_n - f_m\|_\infty + |f_n - f_m|_{Lip} \leq \varepsilon.$$

$$\Rightarrow \forall_{\varepsilon>0} \exists_N \forall_{n,m\geq N} \quad \|f_n - f_m\|_\infty \leq \varepsilon \Rightarrow$$

$C[0,1]$ Banach $\Rightarrow$ $f_n \longrightarrow f$ in $C[0,1]$,

$f \in C[0,1]$.

So f is our candidate for the limit in C_{LIP}.

1: is $f \in C_{LIP}[0,1]$?

It means $\exists_C \ |f(x) - f(y)| \leq C|x-y|$.

But we have

$$|f_n(x) - f_n(y)| \leq |f_n|_{Lip} \, |x-y|$$
$$\leq \left(\sup_n |f_n|_{Lip}\right) |x-y|$$

since Cauchy sequence is always bounded cf. (A3).

We send $n\to\infty$ on the LHS to get

$$|f(x) - f(y)| \leq \left(\sup |f_n|_{Lip}\right) |x-y|.$$

2. Does $\|f_n - f\|_\infty + |f_n - f|_{LIP} \to 0$ as $n \to \infty$.

We know that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$ (this is the way we chose f) so we only need to show $|f_n - f|_{LIP} \to 0$.
So we have to prove

$$\limsup_{n\to\infty} \sup_{x \neq y} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|} \overset{?}{=} 0.$$

We can write it as

$$(*) \quad \limsup_{n\to\infty} \sup_{x \neq y} \limsup_{m\to\infty} \frac{|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))|}{|x-y|} \overset{?}{=} 0.$$

However

$$\sup_{x \neq y} \limsup_{m\to\infty} \dots \leq \limsup_{m\to\infty} \sup_{x \neq y} \dots$$

so

$$(*) \leq \limsup_{n,m\to\infty} \underbrace{\sup_{x \neq y} \frac{|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))|}{|x-y|}}_{= |f_n - f_m|_{LIP}}$$

$$= \limsup_{n,m\to\infty} |f_n - f_m|_{LIP} = 0$$ as $\{f_n\}$ is Cauchy wrt $|\cdot|_{LIP}$. $\blacksquare$

(7) $\rightsquigarrow$ Home work.

D1 This is application of results for L^p with Counting measure. Indeed, then

$$\int |f|^p d\mu = \sum |f_i|^p \quad (\text{for } f=(f_1, f_2, \dots)). \quad \left(\begin{matrix}\mu\text{-counting}\\ \text{measure}\end{matrix}\right) \square.$$

D2 (important!!!) $\rightarrow$ this exercise allows to deduce many properties from considering finite sequences.... For $1 \leq p < \infty$:

$$x - \sum_{i=1}^{n} x_i e_i = (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$$

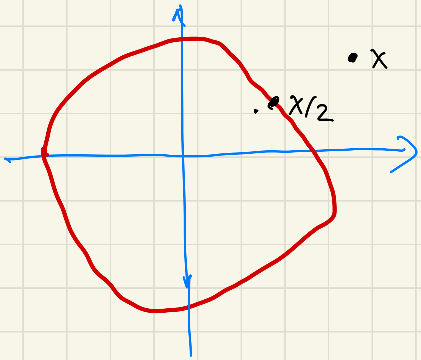
$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \longrightarrow 0$$ as $n \to \infty$ since the series

$\sum_{k=1}^{\infty} |x_k|^p$ is convergent ($\nwarrow$ this is tail of convergent series).

Case $p=\infty$ is NOT true. Take for instance $x=(1, 1, 1, \dots, 1, \dots) \in \ell^\infty$.
(sequence of 1). Then $x - \sum_{i=1}^{n} x_i e_i = (0, 0, 0, \dots, 0, 1, 1, \dots)$
So its norm in ℓ^∞ is 1 for all n.

E1 $(E, \|\cdot\|_E)$ – BS, $C \subset E$ s.t. C is convex, C is open and $0 \in C$.

$$\forall_{x \in E} \quad \rho(x) = \inf\left\{ \alpha > 0 : \frac{x}{\alpha} \in C \right\}.$$



ρ is called <u>Minkowski functional</u>

(a) $\rho(\gamma x) = \gamma\, \rho(x) \quad \forall_{\gamma > 0}$

$$\rho(\gamma x) = \inf\left\{ \alpha > 0 : \frac{\gamma x}{\alpha} \in C \right\} = \inf\left\{ \overset{\beta\cdot\gamma}{\alpha} > 0 : \frac{x}{\underbrace{\alpha/\gamma}_{\beta}} \in C \right\}$$

$$= \inf\left\{ \beta\cdot\gamma : \frac{x}{\beta} \in C,\ \beta\cdot\gamma > 0 \right\}$$

$$= \gamma \cdot \inf\left\{ \beta > 0 : \frac{x}{\beta} \in C \right\} = \gamma \cdot \rho(x).$$

(b) $\underbrace{\rho(x+y)} \leq \rho(x) + \rho(y)$. $\quad \forall\, x, y \in E$

$$\inf\left\{\alpha > 0 : \frac{x+y}{\alpha} \in C\right\} \leq \rho(x) + \rho(y)$$

It would be sufficient to have $\frac{x+y}{\rho(x)+\rho(y)} \in C$.

$$\iff \frac{x}{\rho(x)} \cdot \frac{\rho(x)}{\rho(x)+\rho(y)} + \frac{y}{\rho(y)} \cdot \frac{\rho(y)}{\rho(x)+\rho(y)} \in C.$$

We need $\frac{x}{\rho(x)}, \frac{y}{\rho(y)} \in C$. $\rightarrow$ this is **NOT** true in general

But there is a sequence $\alpha_n \searrow \rho(x)$, $\beta_n \searrow \rho(y)$ s.t. $\frac{x}{\alpha_n}, \frac{y}{\beta_n} \in C$. $\Rightarrow$

$$\Rightarrow \frac{x}{\alpha_n} \cdot \frac{\alpha_n}{\alpha_n+\beta_n} + \frac{y}{\beta_n} \cdot \frac{\beta_n}{\alpha_n+\beta_n} \in C \Rightarrow \frac{x+y}{\alpha_n+\beta_n} \in C.$$

$$\Rightarrow \rho(x+y) \leq \alpha_n + \beta_n \xrightarrow[n\to\infty]{} \rho(x)+\rho(y) \quad \checkmark.$$

(c) There is a constant M s.t. $\rho(x) \le M \|x\|_E$.

$0 \in C$, C is open $\Rightarrow \exists_r \; B(0,r) \subset C$.

$$\forall_x \; \frac{x}{\|x\|_E} \cdot \frac{r}{2} \in C \Rightarrow \frac{x}{\|x\|_E \, 2/r} \in C$$

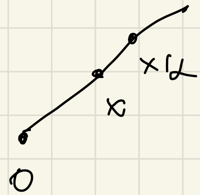
$$\Rightarrow \rho(x) \le \frac{2\|x\|_E}{r} = C \|x\|_E. \quad \left(C = \frac{2}{r}\right).$$

(d) $C = \{x \in E : \rho(x) < 1\}$.

($\Rightarrow$) $x \in C$, $0 \in C$, C is open, $(1+\varepsilon)x \in C$ $\exists_\varepsilon$.

$$\Rightarrow \frac{x}{\frac{1}{1+\varepsilon}} \in C \Rightarrow \rho(x) \le \frac{1}{1+\varepsilon} < 1$$

($\Leftarrow$) $x : \rho(x) < 1 \Rightarrow \exists_{\alpha < 1} \; \frac{x}{\alpha} \in C$



By convexity, $x \in C$. □