Functional Analysis, PS2

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A1

$A = \sup_{\|x\|=1} \|Tx\|_Y$, $B = \sup_{\|x\|\leq 1} \|Tx\|_Y$, $C = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$

We know $B \geq A$, $C = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x\neq 0} \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y =$

$= \sup_{\|x\|=1} \|Tx\|_Y = A.$

Finally $B = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|\leq 1} \left\| \|x\| \, T\left(\frac{x}{\|x\|}\right)\right\|_Y =$

$= \sup_{\|x\|\leq 1} \|x\| \left\| T\left(\frac{x}{\|x\|}\right)\right\| \leq \sup_{\|x\|\leq 1} \left\| T\Big(\underbrace{\frac{x}{\|x\|}}_{\text{norm } 1}\Big)\right\| \leq A.$

A2

(d) ⇒ (c) ⇒ (b) easy

(a) ⇒ (d): $\|T(x-y)\| = \left\|T \frac{(x-y)}{\|x-y\|}\right\| \underbrace{\vphantom{T}}_{\|x-y\|} \leq \|T\| \|x-y\|$

so that T is Lipschitz

(b) ⇒ (a): suppose T is not bdd: $\exists \, x_n$, $\|x_n\|=1$ but $\|Tx_n\| \geq n$

Consider $y_n = \frac{x_n}{n}$ so that $y_n \to 0$ in X. But then

$\|Ty_n\| = \left\|T\frac{x_n}{n}\right\| \geq \frac{1}{n}\|Tx_n\| \geq 1$ so T is not cont.

at 0 ⇒ contradiction.

A3 We need to check that $(\mathcal{L}(X,Y), \|\cdot\|)$ is a normed space where $\|T\| = \sup_{\|x\|_X \le 1} \|Tx\|_Y$.

We first check that $\|T\|$ is operator norm.

1) When $T=0 \Rightarrow \|T\|=0$.

$$\|T\|=0 \Rightarrow \sup_{\|x\|\le 1} \|Tx\|_Y = 0 \Rightarrow \forall_{\|x\|=1} \|Tx\|_Y = 0$$

$$\Rightarrow \forall_{\|x\|=1} Tx = 0 \quad (\text{since } \|\cdot\|_Y \text{ is a norm}).$$

$$\Rightarrow \forall_{x \neq 0} Tx = 0 \quad \left(\text{by scaling } x \to \frac{x}{\|x\|_X}\right).$$

For $x=0$, $Tx=0$ by linearity. $\left(T0 = T(x-x) = Tx - Tx \overset{\text{||}}{0}\right)$.

2) $$\|\alpha T\| = \sup_{\|x\|\le 1} \|(\alpha T)x\|_Y \underset{\substack{\uparrow \\ \|\cdot\|_Y \text{ is a norm}}}{=} |\alpha| \sup_{\|x\|\le 1} \|Tx\|_Y = |\alpha|\, \|T\|.$$

3) Δ: $$\|T+S\| = \sup_{\|x\|\le 1} \|(T+S)x\|_Y \underset{\substack{\uparrow \\ \Delta \text{ for } \|\cdot\|_Y}}{\le} \sup_{\|x\|\le 1} \|Tx\| + \sup_{\|x\|\le 1} \|Ty\|$$

$$= \|T\| + \|S\|.$$

By Δ inequality, $\mathcal{L}(X,Y)$ is a linear space. $\square$.

A4

Note that $\boxed{\|Tx\| \le \|T\| \|x\|}$!!!

Let $\|x\| \le 1$. We have

$$\|\underbrace{STx}_{=S(Tx)}\| \le \|S\| \|Tx\| \le \|S\| \|T\| \|x\| \le \|S\| \|T\|$$

Hence $\|ST\| = \sup_{\|x\| \le 1} \|STx\| \le \|S\| \|T\|$. ✓

General strategy to compute norm of $T: X \to Y$

We need to compute $\|T\| = \sup_{\|x\| \le 1} \|Tx\|$

1) Estimate $\sup_{\|x\| \le 1} \|Tx\| \le C$ with some C.

2) Find x with $\|x\| \le 1$ s.t. $\|Tx\| = C$ or sequence x_n with $\|x_n\| \le 1$ s.t. $\|Tx_n\| \to C$.

Linear functionals form special class of operators.

- We say that φ is a functional (linear functional) on $(X, \|\cdot\|_X)$ if φ is linear and $\varphi: X \to \mathbb{R}$.
- We say that φ is a bounded functional on $(X, \|\cdot\|_X)$ if φ is a linear functional and $\varphi \in \mathcal{L}(X, \mathbb{R})$, i.e. it is bounded.

Remark In finite dim. spaces any linear functional is bounded but in ∞ dim setting, there is ALWAYS a functional which is not bounded. Moreover, even nice functionals like projection on element of basis may be unbounded.

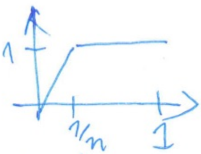
This is discussed in C1, C2, C3.

B1 We need to find $\sup\limits_{\|u\|_\infty \leq 1, u\in E} |\varphi(u)|$. Typically, one first finds bound, then tries to check whether its optimal.

Clearly $|\varphi(u)| \leq \left| \int_0^1 u(t)\, dt \right| \leq \int_0^1 \|u\|_\infty dt = \|u\|_\infty$

so $\sup\limits_{\substack{\|u\|_\infty \leq 1 \\ u\in E}} |\varphi(u)| \leq 1$.

We claim that $\sup\limits_{\|u\|_\infty \leq 1, u\in E} |\varphi(u)| = 1$. Althoug $u(0)=0$ we can approximate with fcns



$= f_n$, $\|f_n\|_\infty \leq 1$, $f_n \in E$.

Note that $\varphi(f_n) = 1 - \frac{1}{n} + \frac{1}{2n} = 1 - \frac{1}{2n} \to 1$.

By the upper bound, $\|\varphi\| = 1$.

There is no $f \in E$, $\|f\|_\infty = 1$ and $\varphi(f) = \|f\|_\infty$ as $f(0)=0$ and)

B2

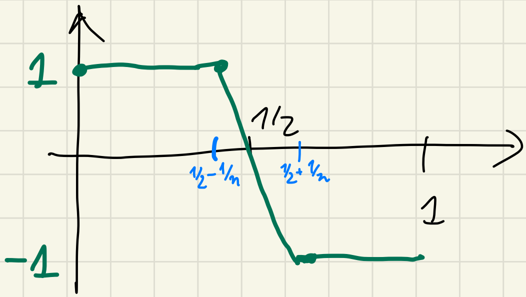
$$\varphi(u) = \int_0^{1/2} u(y)\,dy - \int_{1/2}^{1} u(y)\,dy$$

$\varphi : C[0,1] \to \mathbb{R}$

Let $\|u\|_\infty \leq 1$.

$$|\varphi(u)| \leq \int_0^{1/2} |u(y)|\,dy + \int_{1/2}^{1} |u(y)|\,dy \leq 1.$$

$\Rightarrow \quad \|\varphi\| \leq 1.$



$= u_n \qquad \|u_n\|_\infty \leq 1.$

(even $\|u_n\|_\infty = 1$).

We have $1 \geq \varphi(u_n) \geq 2\left(\frac{1}{2} - \frac{1}{n}\right) = 1 - \frac{2}{n} \to 1$

so $\varphi(u_n) \to 1$. Hence $\|\varphi\| = 1$.

Is there u s.t. $\varphi(u) = \|\varphi\| = 1$? No, it has to be 1 on $[0, \frac{1}{2}]$ and -1 on $[\frac{1}{2}, 1]$ so it cannot be continuous.

$\square$.

B4 $v \in \ell^\infty$ (fixed) $\quad \forall u \in \ell^1 \quad u = (u_1, u_2, \dots) \qquad \varphi(u) = \sum_{i \geq 1} u_i \cdot v_i$

Let u be s.t. $\|u\|_1 \leq 1$. Then

$$|\varphi(u)| = \left| \sum_{i \geq 1} u_i v_i \right| \leq \|v\|_\infty \sum_{i \geq 1} |u_i| \leq \|v\|_\infty$$

$\Rightarrow \|\varphi\| \leq \|v\|_\infty$.

There is a sequence $\{v_{k_m}\}$ s.t. $|v_{k_m}| \to \|v\|_\infty$. We take

$$u^{k_m} = (0, 0, \dots, 0, \operatorname{sgn}(v_{k_m}), 0, 0, \dots)$$

We have $\|u^{k_m}\|_1 = 1$ and $\varphi(u^{k_m}) \to \|v\|_\infty$. It follows that $\|\varphi\| = \|v\|_\infty$.

SPOILER: The problem above shows that $\ell^\infty \subset (\ell^1)^*$. In fact $(\ell^1)^* = \ell^\infty$ and by equality above this is an isometry. Such characterizations of dual spaces are VERY IMPORTANT.

B6 $\varphi(f) = f(\frac{1}{2})$ $\varphi : C[0,1] \to \mathbb{R}$

Let f be s.t. $\|f\|_\infty \leq 1$

$$|\varphi(f)| \leq |f(1/2)| \leq 1 \Rightarrow \|\varphi\| \leq 1$$

Moreover $\varphi(f) = 1$ for $f = 1$. Hence $\|\varphi\| = 1$.

B7 $\varphi(f) = \int_0^1 f(x)\, d\mu(x)$ for μ finite measure on $[0,1]$.

Let f be s.t. $\|f\|_\infty \leq 1$. Then,

$$|\varphi(f)| \leq \int_0^1 |f(x)|\, d\mu(x) \leq \mu([0,1]).$$

Moreover, $\varphi(1) = \mu([0,1])$ so $\|\varphi\| = \mu([0,1])$.

REMARK: We deduce B6 if $\mu = \delta_{1/2}$.

B8 Let $\varphi : C[0,1] \to \mathbb{R}$ be a linear functional s.t.

$\varphi(f) \geq 0$ whenever $f \geq 0$.

Let f be s.t. $\|f\|_\infty \leq 1$. Then $1 - f(x) \geq 0$, $1 + f(x) \geq 0$.

$$\Rightarrow \begin{array}{l} \varphi(f(x)) \leq \varphi(1) \\ -\varphi(f(x)) \leq \varphi(1) \end{array} \Rightarrow -\varphi(1) \leq \varphi(f(x)) \leq \varphi(1)$$

$$\Rightarrow |\varphi(f(x))| \leq |\varphi(1)| \Rightarrow \|\varphi\| \leq \varphi(1).$$

But $\varphi(1) = \varphi(1)$ so that $\|\varphi\| = \varphi(1)$.

We easily deduce B7 as $\varphi(1) = \int_0^1 1 \, d\mu(x) = \mu([0,1])$.

SPOILER: In fact, all nonnegative functionals on $C[0,1]$ are nonneg. measures (Riesz–Markov–Kakutani).

B9 Let $1 \le p < \infty$, $C_d = |B(0,1)|$, $\|f\|_p \le 1$

$$\int_{\mathbb{R}^d} |Tf(y)|^p \, dy = \int_{\mathbb{R}^d} \left[\fint_{B(y,1)} f(x)\, dx \right]^p dy \underset{\substack{\uparrow \\ \text{change} \\ \text{of variables}}}{=} \int_{\mathbb{R}^d} \left[\fint_{B(0,1)} f(x+y)\, dx \right]^p dy$$

$$\underset{\substack{\text{Jensen} \\ +|\int\ldots| \le \int|\ldots|}}{\leq} \int_{\mathbb{R}^d} \fint_{B(0,1)} |f(x+y)|^p \, dx\, dy \underset{\substack{\uparrow \\ \text{Fubini}}}{=} \fint_{B(0,1)} \underbrace{\int_{\mathbb{R}^d} |f(x+y)|^p \, dy}_{x\text{-fixed}} \, dx$$

$$= \fint_{B(0,1)} \|f\|_p^p \, dx = \|f\|_p^p \le 1.$$

Then, let $f_n = \mathbb{1}_{B(0,n)} / |B(0,n)|^{1/p} = \left(\frac{1}{n^d C_d} \right)^{1/p} \mathbb{1}_{B(0,n)}$

$$\int_{\mathbb{R}^d} |Tf_n(y)|^p \, dy = \frac{1}{n^d C_d} \int_{\mathbb{R}^d} |\underbrace{T \mathbb{1}_{B(0,n)}}_{\ge \mathbb{1}_{B(0,n-2)}}|^p \, dy \ge \frac{(n-2)^d C_d}{n^d C_d}$$

$$= \left(1 - \frac{2}{n}\right)^d \longrightarrow 1 \text{ as } n \to \infty.$$

For $p = \infty$

$$\|Tf\|_\infty \le \fint_{B(y,1)} \|f\|_\infty \le \|f\|_\infty \Rightarrow \|T\| \le 1$$

and this is attained for $f = 1$.

(B14) $T : (C^1[0,1], \|\cdot\|_\infty) \longrightarrow (C[0,1], \|\cdot\|_\infty)$.

only norm of the function

$Tf = f'$ (if $f \in C^1[0,1]$, $f \in C[0,1] \Rightarrow$ well def.)

Is $T \in \mathcal{L}((C^1[0,1], \|\cdot\|_\infty), (C[0,1], \|\cdot\|_\infty))$?

Suppose it is. Then, there is a constant C s.t.

$$\|Tf\|_\infty \leq C \quad \forall_{f \in C^1[0,1]} \text{ s.t. } \|f\|_\infty \leq 1.$$

$$(*) \quad \|f'\|_\infty \leq C \quad \forall_{f \in C^1[0,1]} \text{ s.t. } \|f\|_\infty \leq 1.$$

Let $f_n = x^n$. Then $f_n' = nx^{n-1}$ so that

$\|f_n\|_\infty = 1$, $\|f_n'\|_\infty = n$. It follows from $(*)$ that

$$n \leq C \quad \forall_{n \in \mathbb{N}} \Rightarrow \text{CONTRADICTION.}$$

C1 X - fin. dim, X has basis $\{e_1, \dots, e_n\}$, $X \simeq \mathbb{R}^n$.

For $x \in X$ we write $x = \sum_{i=1}^{n} x_i e_i$. We can take ℓ^1 norm on X

i.e. $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ as all norms on $\mathbb{R}^n$ (fin. dim. spaces are equivalent). (Analysis II.1)

Then, for $a_i := \varphi(e_i)$,

$$\varphi(x) = \varphi\left(\sum x_i e_i\right) = \sum x_i \varphi(e_i) = \sum x_i a_i$$

$$\Rightarrow |\varphi(x)| \leq \sup_{1 \leq i \leq n} |a_i| \sum_{i=1}^{n} |x_i| \leq \sup_{1 \leq i \leq N} |a_i| \cdot \|x\|_1.$$

C2 X - inf. dim, X has basis (possibly uncountable) $\{e_i\}_{i \in I}$ s.t.

- $\forall_{x \in X}$ $\exists$ finitely many $\{e_j\}_{j=1}^{N_x}$ $\exists!$ $\{a_j\}_{j=1}^{N_x} \subset \mathbb{R}$ $x = \sum_{j=1}^{N_x} a_j e_j$ (span)

- $\forall_{\{e_j\}_{j=1}^{N} \text{ finite}}$ $\sum_{j=1}^{N} a_j e_j = 0 \Rightarrow a_j = 0$. (linear independence)

Take countable subset, assume $\|e_i\| = 1$ and define $\varphi(e_i) = i$. Then $\|\varphi\| > i$ $\forall_i$. It follows that $\|\varphi\| = \infty$.

C3 $\left(\mathcal{P}[0,1], \|\cdot\|_1\right)$ $\quad$ (i.e. $\|f\|_1 = \int |f|$).

Let $\varphi_0(a_0 + a_1 x + \dots + a_n x^n) = a_0$ (this may be seen as a projection functional). We gonna prove that φ is not cont.

$$\varphi_0\left((x-1)^n\right) = \begin{cases} 1, & n \text{ even}, \\ -1, & n \text{ odd}. \end{cases}$$

On the other hand, $(x-1)^n \to 0$ in $\|\cdot\|_1$ because

$$\int_0^1 (x-1)^n \, dx \to 0 \text{ by dominated convergence}$$

So we have $f_n \to 0$ in $(\mathcal{P}[0,1], \|\cdot\|_1)$ but $\varphi(f_n) \not\to 0$.

More generally, if k is fixed, φ_k is projection

$$\varphi_n(a_0 + \dots + a_k x^k + \dots + a_n x^n) = a_k,$$

φ_k is also discontinuous on $(\mathcal{P}[0,1], \|\cdot\|_1)$. Indeed,

$$\varphi_k\left(x^k (x-1)^n\right) = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd}. \end{cases}$$

Moreover, $\int x^k (x-1)^n \to 0$ by dominated convergence.

Again, we have $f_n \to 0$ in $(\mathcal{P}[0,1], \|\cdot\|_1)$ but $\varphi_k(f_n) \not\to 0$ $\blacksquare$

D1 $(X, \|\cdot\|_X)$ normed space.

We know that if $(Y, \|\cdot\|_Y)$ is Banach, then $\mathcal{L}(X,Y)$ is also Banach (with operator norm) (Lecture).

Hence, $X^* = \mathcal{L}(X, \mathbb{R})$ is Banach.

D3 $(X, \|\cdot\|_X)$ – normed
$(Y, \|\cdot\|_Y)$ – Banach $\Rightarrow \mathcal{L}(X,Y)$ Banach.

In Banach spaces $\sum \|x_k\| < \infty \Rightarrow \sum x_k < \infty$.

We want $\sum \frac{T^k}{k!} < \infty$. We study $\|T^k\|$.

$$\|T^k\| \leq \|T\| \, \|T^{k-1}\| \leq \dots \leq \|T\|^k.$$

So $\left\|\frac{T^k}{k!}\right\| \leq \frac{\|T\|^k}{k!}$. We have $\sum \frac{\|T\|^k}{k!} = e^{\|T\|} < \infty$.

It follows that $\sum \frac{T^k}{k!}$ converges in $\mathcal{L}(X,Y)$.

COMMENT: This is the way to solve PDEs of the form $u_t = \Delta u$, $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ written as $u_t = Au$ for some operator A. See Hille-Yosida theorem

D4 Ass D3.