Functional Analysis, PS3

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A1

Suppose X has countable Hamel basis $\{e_i\}_{i\in\mathbb{N}}$.

Write $X_n = \text{span}\{e_1, e_2, \dots, e_n\}$. We have

→ X_n is closed as this is finite dim. space.

(Indeed, let $x^n = \sum_{i=1}^{d} x_i^n e_i$, $x_i^n \in \mathbb{R}$, $x^n \to x$ $\Rightarrow$ by equivalence of norms $\{x_i^n\}_i$ are Cauchy in $\mathbb{R}$ and have limits x_i. We claim that $x = \sum_{i=1}^{d} x_i e_i$. Indeed, by equivalence $\|x - x^n\| \leq \sum_{i=1}^{d} |x_i - x_i^n| \to 0$. By uniqueness of limits $x = \sum_{i=1}^{d} x_i e_i$.)

→ X_n has empty interior. Suppose, that there is $x \in X_n$ and $r > 0$ s.t. $B(x,r) \subset X_n$. Then, $x + e_{i+1}\frac{r}{2}$ is in $B(x,r)$ but $x + e_{i+1}\frac{r}{2} \notin X_n$.

It follows that $\bigcup X_n$ has empty interior in X. But $X = \bigcup X_n$ by assumption. ■

A2, A3

This spaces have countable Hamel basis $\{e_i\}$ s.t. $e_i = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ i\text{th pos.}}}{1}, 0, \dots)$ and

$\{x^k\}$.

B1 $\Uparrow$ Counterexample for Banach-Steinhaus.

B2 BS: $T_n : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$

$$\forall_{x \in X} \quad \sup_{n \in \mathbb{N}} \|T_n x\|_Y < \infty \Rightarrow \sup_{n \in \mathbb{N}} \|T_n\|_Y < \infty$$

Equivalently (or more precisely)

$$\forall_x \exists_{C_x} \quad \sup_{n \in \mathbb{N}} \|T_n x\|_Y \leq C_x \Rightarrow \exists_C \sup_{n \in \mathbb{N}} \sup_{\|x\| \leq 1} \|T_n x\|_Y \leq C$$

Problem $\forall_{g \in L^2(0,1)} \quad \int_0^1 f_n g \, dx \longrightarrow C_g \in \mathbb{R}$

$$T_n : L^2(0,1) \longrightarrow \mathbb{R} \qquad T_n(g) = \int_0^1 f_n(x) g(x) \, dx.$$

$$\forall_g \exists_{C_g} \sup_{n \in \mathbb{N}} |T_n g| \leq C_g \iff$$

$$\forall_{g \in L^2(0,1)} \exists_{C_g} \sup_{n \in \mathbb{N}} \left| \int_0^1 f_n(x) g(x) \, dx \right| \leq C_g.$$

(This is true because convergent sequence is bounded)

Thanks to Banach-Steinhaus

$$\exists_C \sup_{n\in\mathbb{N}} \sup_{\|g\|_2\le 1} |T_n g| \le C$$

$$\Longleftrightarrow \exists_C \sup_{n\in\mathbb{N}} \underline{\sup_{\|g\|_2\le 1} \left| \int_0^1 f_n(x)\, g(x)\, dx \right|} \le C$$

Take $g = f_n / \|f_n\|_2$ to get $\exists_C \sup_{n\in\mathbb{N}} \|f_n\|_2 \le C$. $\square$

B3 $(X, \|\cdot\|_X)$ – Banach space.

Let $A \subset X^*$ be such that $\forall_{x\in X} \{\varphi(x) : \varphi \in A\}$ is bounded. Prove that A is bounded in X^*.

1) $A \subset X^*$. X^* is the space of bounded functionals on X equipped with the operator norm.

2) We know that $\forall_{x\in X} \{\varphi(x) : \varphi \in A\}$ is bounded. This is subset of $\mathbb{R}$.

3) We have to prove that A is bdd in X^*, i.e:

$$\sup_{\varphi\in A} \|\varphi\| < \infty.$$

$$\forall_x \exists_{C_x} \sup_{n\in\mathbb{N}} \|T_n x\|_Y \le C_x \Rightarrow \exists_C \sup_{n\in\mathbb{N}} \sup_{\|x\|\le 1} \|T_n x\|_Y \le C$$

$(T_n : X \to Y)$.

As $\forall_x$ $\{\varphi(x) : \varphi \in A\}$ is bounded we take

$$T_\varphi x := \varphi(x) \quad , \quad \varphi \in A, \; T_\varphi : X \to \mathbb{R}.$$

We check assumptions

$\forall_x \exists_{C_x} |T_\varphi x| \le C_x$ ← follows from . Hence,

Banach-Steinhaus implies

$$\exists_C \sup_{\varphi\in A} \underbrace{\sup_{\|x\|\le 1} |\varphi(x)|}_{= \|\varphi\|} \le C$$

$= \|\varphi\|$ ← operator norm.

$$\Rightarrow \exists_C \sup_{\varphi\in A} \|\varphi\| \le C.$$

B4 $\Uparrow$.

B5 $X = (P[0,1], \|\cdot\|_1)$ – not a Banach space.

Consider
$$\beta(f,g) = \int f(t)g(t)\,dt$$

For fixed f $g \mapsto B(f,g)$ is continuous on X:

$$|B(f,g)| \le \|f\|_\infty \int |g(t)|\,dt \le \|f\|_\infty \|g\|_1 .$$

Similarly, for fixed g, $f \mapsto B(f,g)$ is cont. on X.

<u>But</u> $(f,g) \mapsto B(f,g)$ is not continuous.

$$|B(f,g)| \le \int |f(t)|\,dt \int |g(t)|\,dt$$

i.e. $\left|\int fg\right| \le \int |f| \int |g|$

take $f=g$ $\quad \left|\int f^2\right| \le \left(\int |f|\right)^2 .$

Consider $f = x^n$ $\quad \left|\int x^{2n}\right| \le \left(\int x^n\right)^2$

i.e. $\frac{1}{2n+1} \le \frac{1}{(n+1)^2} \Rightarrow \frac{(n+1)^2}{2n+1} \le 1$

contradiction.

B7 • $\{T_n\} \subset \mathcal{L}(X, Y)$

• $\forall_x$ $T_n x$ converges to some Tx

$\Rightarrow$ T is a bounded operator.

STEP 1: $\sup_n \|T_n\| < \infty$.

$\forall_x$ $\{T_n x\}_n$ is a bounded sequence in Y. From Banach-Steinhaus $\sup_n \|T_n\| < \infty$.

STEP 2: $\|Tx\| \leq \|(T - T_n)x\| + \|T_n x\|$

Apply liminf to get for $\|x\| \leq 1$.

$$\|Tx\| \leq \liminf_{n\to\infty} \|T_n x\| \leq \liminf_{n\to\infty} \|T_n\|$$

Take sup on the (LHS) to get:

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\| < \infty.$$

$\square$.

B8

$T_n : X \to X \qquad T_n x = (x_1, 2x_2, \dots)$

(a)
$$\|T_n x\| = \sup_{1 \le k \le n} |k x_k| \le n \|x\|_\infty$$

$\Rightarrow T_n \in \mathcal{L}(X, X)$.

(b) As x has only finitely many non zero terms, $T_n x$ becomes constant sequence.

(c) $T : X \to X$ is not bounded.

Suppose $\exists_C \quad \|Tx\| \le C\|x\| \quad \forall_x$.

Consider $x = (0, 0, \dots, 0, 1, 0, \dots)$

$$Tx = (0, 0, \dots, 0, n, 0, \dots)$$

$\Rightarrow \quad n \le C \quad \forall_{n \in \mathbb{N}} \Rightarrow$ contradiction

(d) X has countable Hamel basis.