Functional Analysis, PS4

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Closed graph theorem

(Used to prove boundedness of operators which are not defined by explicit formulas).

CGT: $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ linear

Banach spaces

$T \in \mathcal{L}(X, Y) \iff G(T) = \{(x, Tx) : x \in X\} \subset X \times Y$ is closed.

$\iff \forall \{x_n\} \subset X$ s.t. $x_n \to x$, $Tx_n \to y$ we have $Tx = y$.

A1 $T \in \mathcal{L}(X, Y)$, T is bijective. $\overset{?}{\Rightarrow} T^{-1} \in \mathcal{L}(X, Y)$.

<u>Sol</u>: $G(T^{-1}) = \{(y, T^{-1}y) : y \in Y\}$

Let $y_n \to y \in Y$, $T^{-1}y_n \to z \in X$. We have to check that $z = T^{-1}y$, Let $y_n \to y \in Y$, $T^{-1}y_n \to z \in X$. We have to check that $z = T^{-1}y$.

$$\begin{array}{ccc} T(T^{-1}y_n) & \longrightarrow & Tz \quad \text{since } T \in \mathcal{L}(X,Y). \\ \| & & \\ y_n & \longrightarrow & y \end{array}$$

$$\Rightarrow \; Tz = y \;\Rightarrow\; z = T^{-1}y \quad (T \text{ is bijective}). \qquad \blacksquare$$

A2 $(X, \|\cdot\|)$ – BS $\quad T: X \to X^*$ s.t.

$$\forall_{x,y \in X} \quad \underbrace{\underbrace{(Tx)}_{\in X^*}(y)}_{\in \mathbb{R}} = (Ty)(x) \qquad (\text{self-adjoint})$$

operator norm

Prove that $T \in \mathcal{L}(X, X^*)$. As $(X^*, \|\cdot\|)$ is Banach space, we can use closed graph theorem.

$$G(T) = \{(x, Tx) : x \in X\} \subset X \times X^*$$

We need to check that $G(T)$ is closed in $X \times X^*$.

i.e. $x_n \longrightarrow x$ in $(X, \|\cdot\|_X)$

$Tx_n \to y$ in $(X^*, \|\cdot\|)$ operator norm

$\overset{?}{\Rightarrow}$ $y = Tx$. (i.e. $\forall_{z \in X}$ $y(z) = (Tx)(z)$).

We need to use condition $(Tu)(v) = (Tv)u$ $\forall_{u,v \in X}$.

Let $u = x_n$, $v = z$, z arbitrary.

$$(Tx_n)(z) = (Tz)x_n$$

<u>Limit of $(Tx_n)(z)$:</u> $Tx_n \to y$ in $(X^*, \|\cdot\|)$, i.e.

$$\|Tx_n - y\| \longrightarrow 0 \Rightarrow \sup_{\|z\|_X \leq 1} (Tx_n - y)(z) \longrightarrow 0$$

$$\Rightarrow \forall_{\|z\| \leq 1} (Tx_n)(z) \to y(z). \Rightarrow \forall_z (Tx_n)(z) \to y(z)$$

(by scaling).

Hence $(Tx_n)(z) \to y(z)$.

<u>Limit of $(Tz)(x_n)$</u> $T_z \in X^*$ - continuous. As $x_n \to x$,

$(Tz)(x_n) \longrightarrow (Tz)(x)$ $\Rightarrow$ $(Tz)(x) = y(z)$ $\Rightarrow$

$$(Tz)(x) = y(z) \Rightarrow (Tx)(z) = y(z) \quad \forall_z$$

$$\Rightarrow Tx = y \text{ in } X^*. \qquad \blacksquare$$

A3 $\Uparrow$ (next week; with hint).

$$T: X \to X^* \qquad (Tx)(x) \geq 0 \Rightarrow T \in \mathcal{L}(X, X^*).$$

A7

$$(*)\begin{cases} x_f^{(2020)} + t x_f^{(2019)} + \dots + t^{2019} x_f^{(1)} + t^{2020} x_f = f(t) \\ x_f^{(i)} = 0 \qquad 0 \leq i \leq 2019. \end{cases}$$

$$T: C[0,1] \to C^{2020}[0,1] \qquad Tf = x_f$$

To study $(*)$ we introduce

$$y_{1,f}(t) = x_f(t)$$

$$y_{2,f}(t) = x_f^{(1)}(t)$$

$$\vdots$$

$$y_{2019,f}(t) = x_f^{(2018)}(t)$$

$$y_{2020,f}(t) = x_f^{(2019)}(t)$$

Then equation can be written as

$$\begin{cases} y'_{2020,f} + t\, y_{2020,f} + t^2 y_{2019,f} + \dots + t^{2020} y_{1,f} = f \\ y'_{2019,f} - y_{2020,f} = 0 \\ y'_{2018,f} - y_{2019,f} = 0 \\ \dots \\ y'_{1,f} - y_{2,f} = 0 \end{cases}$$

This can be written in the matrix form

$$\frac{d}{dt}\vec{y} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \mathbf{0} & \\ & & \ddots & 1 & & \\ & & & \ddots & 0 & 1 \\ & & & & 0 & 1 \\ -t^{2020} & -t^{2019} & \cdots & \cdots & -t^2 & -t \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}.$$

By Picard-Lindelöf, it has a unique solution so T is well-defined.

$T \in \mathcal{L}(C[0,1], C^{2020}[0,1])$?

Let $(f_n, Tf_n) \to (f, y)$ in $C[0,1] \times C^{2020}[0,1]$.

$$\begin{cases} x_{f_n}^{(2020)} + t\, x_{f_n}^{(2019)} + \dots + t^{2019} x_{f_n}^{(1)} + t^{2020} x_{f_n} = f_n(t) \\ x_{f_n}^{(i)} = 0 \qquad 0 \leq i \leq 2019. \end{cases}$$

$\Rightarrow$

$$\begin{cases} y^{(2020)} + t\, y^{(2019)} + \dots + t^{2019} y^{(1)} + t^{2020} y = f(t) \\ \qquad\qquad y_f^{(i)} \end{cases}$$

as uniform convergence implies pointwise convergence. Uniqueness theorem implies $Tf = y$. We apply closed graph theorem to conclude the proof.

C4 Suppose there is another norm on $C[0,1]$ which makes it Banach and implies ptwise convergence.

We write $X = (C[0,1], \|\cdot\|_\infty)$ — standard $C[0,1]$ space

$Y = (C[0,1], \|\cdot\|_A)$ — $C[0,1]$ with $\|\cdot\|_A$ norm we study

By assumption, X, Y are Banach spaces. Consider $T: X \to Y$, $T = Id$.

We study closedness of graph of T. $G(T) = \{(f, f) \in X \times Y\}$.

Suppose $f_n \to f$ in X. We need $f = g$.

$f_n \to g$ in Y.

Since $\underset{(\text{in } X)}{f_n \to f} \Rightarrow f_n \to f$ pointwise

Since $f_n \to g$ in $Y \Rightarrow f_n \to g$ pointwise (as we assumed that convergence in $\|\cdot\|_A$ implies ptwise convergence). Hence, $f = g$.

$\Rightarrow T$ is bounded by CGT and $\|f\|_A \leq C\|f\|_\infty$

Similar argument shows $\|f\|_\infty \leq \tilde{C}\|f\|_A$. $\square$.

C5 Exactly the same like C3 but this time convergence of subsequence is used.

$\square$.

B1 • $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ ← both Banach spaces.

• $\|x\|_1 \leq C \|x\|_2$.

$$\overset{?}{\Longrightarrow} \underset{\tilde{C}}{\exists} \; \|x\|_2 \leq C \|x\|_1 \quad (\text{i.e. } \|\cdot\|_1, \|\cdot\|_2 \text{ are equivalent})$$

Proof: consider $T : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ given by $Tx = x$, i.e. T is an identity operator.

By assumption T is bounded. Clearly, identity operator is bijective. It follows that T^{-1} is bounded, i.e.

$$\|T^{-1} x\|_2 \leq C \|x\|_1 \text{ for some } C \quad \text{i.e.}$$

$$\|x\|_2 \leq C \|x\|_1 \text{ for some}$$

This is mostly used to prove that $(X, \|\cdot\|_X)$ is not a Banach space.

B2 $C[0,1]$ with $L^p(0,1)$ $(1 \le p < \infty)$ norm is not a Banach space. Clearly

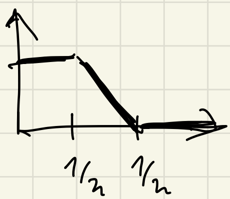
$$\|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p} \le \|f\|_\infty$$

If $(C[0,1], \|\cdot\|_p)$ was a Banach space, there would be a constant s.t.

$$\|f\|_\infty \le C\,\|f\|_p \qquad \forall f \in C[0,1]$$

i.e. $\|f\|_\infty \le C\left(\int |f(t)|^p\right)^{1/p}$. $(*)$

Take $f_n = 1$



$$\int_0^1 |f_n(t)|^p \le \frac{2}{n}$$

$(*)$ implies $1 \le C \cdot \left(\frac{2}{n}\right)^{1/p}$. Send $n \to \infty$ to get contradiction.

B3 Prove that $(\ell^1, \|\cdot\|_\infty)$ is not a Banach space.

Suppose that $(\ell^1, \|\cdot\|_\infty)$ is Banach. We also know that $(\ell^1, \|\cdot\|_1)$ is Banach. Moreover, for $x \in \ell^1$

$$\|x\|_\infty \leq \|x\|_1$$

By (B1) we have $\|x\|_1 \leq C\|x\|_\infty$ $\exists_C$ $\forall x \in \ell^1$.

Consider $x_n = \underbrace{(1, 1, \dots, 1}_{n}, 0, 0, \dots)$. We have

$n \leq C$ $\forall_{n \in \mathbb{N}}$. $\Rightarrow$ CONTRADICTION.

B5

$$I + c_1 A + c_2 A^2 + \dots + c_n A^n = 0$$

A is injective: $Ax = 0 \Rightarrow x = 0$ (we apply equality above to x and get $x = 0$).

A is surjective: given y find x s.t. $Ax = y$.

$$y + c_1 Ay + c_2 A^2 y + \dots c_n A^n y = 0$$

$$y = A[-c_1 y - c_2 Ay + \dots - c_n A^{n-1} y]. \quad \square$$