Functional Analysis, PS 6

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A1 $H = H^*$

$$\forall_{\varphi \in H^*} \ \exists!_{u_\varphi} \quad \varphi(h) = \langle u_\varphi, h \rangle$$

$T : H^* \to H$ def. with $T(\varphi) = u_\varphi$.

A2 $(L^p)^* = L^q \qquad 1 \leq p < \infty \ , \ \frac{1}{p} + \frac{1}{q} = 1.$

$$\forall_{\varphi \in (L^p)^*} \ \exists_{g_\varphi \in L^q} \quad \varphi(f) = \int f g_\varphi$$

$$\| \varphi \|_{(L^p)^*} = \| g_\varphi \|_q$$

A3 First, $\varphi \in (L^p)^*$ $1 < p < \infty$?

It is sufficient to check if $e^{-t} \in L^{p'}$?

$$\| e^{-t} \|_{p'}^{p'} = \int_{\mathbb{R}^+} e^{-tp'} \, dt = \frac{1}{p'} \Rightarrow \| \varphi \| = \left(\frac{1}{p'} \right)^{1/p'}$$

$\underline{p=1}$: we can still use duality

$$\|\varphi\| = \|e^{-t}\|_\infty = 1$$

$\underline{p=\infty}$: we have to compute this directly. When $\|f\|_\infty \leq 1$ we have

$$|\varphi(f)| \leq \int_{\mathbb{R}^+} e^{-t} = 1.$$ Take $f=1$ to get $\|\varphi\| = 1$.

A4 $(\mathbb{R}^N)^* = \mathbb{R}^N$

$\varphi \in (\mathbb{R}^N)^* \qquad \varphi(x) = \varphi(x_1, x_2, \dots, x_n) =$

$$= \varphi\left(\sum x_i e_i\right) = \sum x_i \varphi(e_i)$$

$$\varphi \longmapsto (\varphi(e_1), \dots, \varphi(e_n)) \in \mathbb{R}^N.$$

A5 $\varphi \in (X \times \mathbb{R})^*$

$$\varphi(u,a) = \varphi(u,0) + \varphi(0,a) =$$

$u \in X$, $a \in \mathbb{R}$

$$= \underbrace{\varphi(u,0)}_{\text{some } X^*} + a \underbrace{\varphi(0,1)}_{\text{some constant}}$$

each functional on $(X, \mathbb{R})^*$ is of the form

$$\varphi(u,a) = \psi(u) + a\,c \qquad \exists \psi \in X^* \quad \exists c \in \mathbb{R}$$

A6 Let $0 < p < 1$. We consider $L^p(0,1)$ with metric

$$d(f,g) = \int |f(x) - g(x)|^p \, dx$$

Suppose there is continuous $\varphi : L^p(0,1) \to \mathbb{R}$, $\varphi \neq 0$.

1. There is f s.t. $|\varphi(f)| \geq 1$. (indeed, φ is surjective).

2. For such f there is s such that

$$\int_0^s |f(x)|^p = \frac{1}{2} \int_0^1 |f(x)|^p > 0.$$

Let $g_1 = f \mathbb{1}_{[0,s]}$, $g_2 = f \mathbb{1}_{(s,1]}$ so that

$f = g_1 + g_2$, $|f|^p = |g_1|^p + |g_2|^p$ and

$\int |g_1|^p = \int |g_2|^p = \frac{1}{2} \int |f|^p$. Moreover

there is $i=1$ or $i=2$ s.t. $|\varphi(g_i)| \geq \frac{1}{2}$.

Define $f_1 = 2g_i$.

$$|\varphi(f_1)| \geq 1, \quad \int_0^1 |f_1(x)|^p = 2^p \int_0^1 |g_i| =$$

$$= 2^p \frac{1}{2} \int_0^1 |f|^p = 2^{p-1} \int_0^1 |f|^p.$$

By induction, we construct $\{f_n\}$ s.t. $|\varphi(f_n)| \geq 1$ but $d(f_n, 0) \longrightarrow 0$. Contradiction.

A7 $\underbrace{\underbrace{(J\underbrace{x}_{\in E})}_{\in E^{**}}(\underbrace{\varphi}_{\in E^*})}_{\in \mathbb{R}} = \underbrace{\underbrace{\varphi}_{\in E^*}(\underbrace{x}_{\in E})}_{\in \mathbb{R}}$

(A) J is well defined (i.e. for $x \in E$, $Jx \in E^{**}$).

$$\|Jx\|_{E^{**}} = \sup_{\|\varphi\|_{E^*} \le 1} |Jx(\varphi)| = \sup_{\|\varphi\|_{E^*} \le 1} |\varphi(x)|$$

$$= \|x\|_E < \infty.$$

This also proves injectivity.

(B) We need to prove J is surjective.

Let $\Psi \in H^{**}$.. $H = H^*$ in the sense of $P: H \to H^*$

$\Psi(P(x)) \in H^* \Rightarrow \exists_{a_\Psi} \quad \Psi(P(x)) = (a_\Psi, x)$.

$\Psi(\varphi) = \Psi(P^{-1}(\varphi)) = (a_\Psi, P^{-1}(\varphi)) = \varphi(a_\Psi)$.

where we used $\psi(x) = (x, P^{-1}(\psi))$.

(c) J is isomorphism between E and E^{**}.
Both J, J^{-1} is bounded by $\|Jx\| = \|x\|$.

Recall that E^{**} is always Banach. It follows that E is Banach too.

Because: $(x_n)_{n\geq 1}$ Cauchy in E $\Rightarrow$ $(Jx_n)_{n\geq 1}$ is Cauchy in E^{**} $\Rightarrow$ $(Jx_n)_{n\geq 1}$ converges in E^{**} to some $y = Jx$ as J is isomorphism. Finally

$$\|x_n - x\| = \|Jx_n - Jx\| \longrightarrow 0. \Rightarrow x_n \to x.$$

$\square$.

A8 $T: \ell^1 \to (c_0)^*$

$$(Ty)(x) = \sum x_i \cdot y_i$$

- well-defined $y \in \ell^1 \Rightarrow Ty \in (c_0)^*$

Indeed, $|(Ty)(x)| \leq |\sum x_i \cdot y_i| \leq \|y\|_1 \|x\|_\infty$

$\Rightarrow \|Ty\| \leq \|y\|_1$.

- injective: $Ty = 0$ for some $y \in \ell^1$, i.e.

$\forall_{x \in c_0}$ $\sum x_i \cdot y_i = 0$. Take $x = e^i$ to get $y = 0$.

- surjective: given $\varphi \in (c_0)^*$

We claim that $y = (\varphi(e_1), \varphi(e_2), \dots)$ works, i.e. $Ty = \varphi$.

First, when $x \in \operatorname{span}\{e_1, e_2, \dots\}$ = finite combinations of $e_1, e_2, \dots$ we have for some N

$$(T_y)(x) = \sum_{i=1}^{N} x_i y_i = \sum_{i=1}^{N} x_i \varphi(e_i) = \varphi\left(\sum_{i=1}^{N} x_i e_i\right)$$

$$= \varphi(x) \quad \text{where } x = \sum_{i=1}^{N} x_i e_i.$$

So $T_y = \varphi$ on $\operatorname{span}\{e_1, e_2, \dots\}$. But $\overline{\operatorname{span}\{e_1, e_2, \dots\}} = c_0$ and so, $T_y = \varphi$.

- We have $\|T_y\|_{(c_0)^*} \leq \|y\|_1$

To get equality consider $x^n \in c_0$ given by

$$x^n = (\operatorname{sgn} y_1, \operatorname{sgn} y_2, \dots, \operatorname{sgn} y_n, 0, 0, \dots)$$

$$(T_y)(x^n) = \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} |y_i|$$

$$\Rightarrow \|T_y\|_{(c^0)^*} \geq \sum_{i=1}^{n} |y_i| \Rightarrow \|T_y\|_{(c^0)^*} \geq \|y\|_1$$

Therefore $\|T_y\|_{(c^0)^*} = \|y\|_1$.

$\square$

B1 Consider $p(x) = \|x\| \, \|g\|_{M^*}$ in analytic form of Hahn-Banach Thm. Theorem.

Clearly $p(x+y) \leq p(x) + p(y)$, $p(tx) = t p(x)$ $\forall_{t>0}$. $\Rightarrow \exists_{f: X \to \mathbb{R}}$

$\forall_{x \in X}$ $f(x) \leq \|x\| \cdot \|g\|_{M^*}$. We want to bound $\sup\limits_{\|x\| \neq 0} \frac{|f(x)|}{\|x\|}$.

$$\frac{|f(x)|}{\|x\|} \leq \begin{cases} \text{if } f(x) \text{ is positive} : \leq \|g\|_{M^*} \\ \text{if } f(x) \text{ is negative}: -\frac{f(x)}{\|x\|} = \frac{f(-x)}{\|x\|} = \frac{f(-x)}{\|-x\|} \leq \|g\|_{M^*} \end{cases}$$

As $f = g$ on M, $\|f\|_{X^*} = \|g\|_{M^*}$.

B2 We define φ_{x_0} on $\mathrm{lin}(\{x_0\})$ with

$\varphi_{x_0}(tx_0) = t\|x_0\|^2$. We have $\|\varphi_{x_0}\| = \|x_0\|$.

Then, we extend φ_{x_0} to the whole of X.

To prove $\|x_0\| = \sup\limits_{\|f\| \leq 1} f(x_0)$ we note that we have inequality "$\geq$". To get equality we consider $f = \frac{\varphi_{x_0}}{\|x_0\|}$

which yields $f(x_0) = \frac{\varphi(x_0)}{\|x_0\|} = \frac{\|x_0\|^2}{\|x_0\|} = \|x_0\|$. $\square$.

B3 Follows from B2 as $\| x_1 - x_2 \| = \sup_{\|\varphi\| \le 1} \varphi(x_1 - x_2)$

$= 0. \implies x_1 = x_2.$

B4 $(E, \|\cdot\|_E)$ – Banach space, $A \subset E$

$\forall_{f \in E^*}$ $f(A)$ is bounded in $\mathbb{R}$. Prove that A is bounded in E.

$$T_x(f) = f(x) \qquad (x \in A, \quad T_x : E^* \to \mathbb{R})$$

T_x is bdd as $|T_x(f)| \le |f(x)| \le \|f\| \|x\|.$

For each f $\sup_{x \in A} |T_x(f)| = \sup_{x \in A} |f(x)| < \infty$

$$\underset{\text{B-S}}{\Longrightarrow} \sup_{\|f\| \le 1} \sup_{x \in A} |T_x(f)| \le C \quad \exists_C$$

$$\Downarrow$$

$$\sup_{x \in A} \underbrace{\sup_{\|f\| \le 1} |f(x)|}_{\|x\|} \le C$$

$\square$.

B5 $1 \leq p \leq \infty$ $\quad \|f\|_p = \sup_{\|g\|_q \leq 1} \int f(x) g(x) \, d\mu(x)$

We know that $\|x\| = \sup_{\substack{\|\varphi\| \leq 1 \\ \varphi \in E^*}} \varphi(x)$ for $x \in E$.

We take $E = L^p$ so that $E^* = L^q$ for $1 \leq p < \infty$.

For $p = \infty$ we use that $\|\varphi\| = \sup_{\|x\| \leq 1} \varphi(x)$ for $\varphi \in E^*$.

B6 $L^1(0,1) \subset (L^\infty(0,1))^*$ but $L^1(0,1) \neq (L^\infty(0,1))^*$.

$$L^1(0,1) \ni f \longmapsto \left[L^\infty(0,1) \ni g \longmapsto \int_0^1 fg \right]$$

$$(\Phi(f))(g) = \int_0^1 fg.$$

- $\Phi(f) \in (L^\infty(0,1))^*$ for all $f \in L^1(0,1)$?

  Yes, it is linear. And $|\Phi(f)(g)| \leq \|f\|_1 \|g\|_\infty$

  $\Rightarrow \|\Phi(f)\| \leq \|g\|_\infty \Rightarrow$ well-def.

- there are functionals on $L^\infty(0,1)$ that cannot be described in this way.

  Let $E \subset C[0,1]$ be the space of functions s.t. $f(1/2)=0$ (this is btw Banach space with $\|\cdot\|_\infty$ norm).

  On $C[0,1]$ we define functional $\varphi(f) = f(1/2)$, $\varphi \in (C[0,1])^*$. We extend φ to $L^\infty(0,1)$ using analytic Hahn-Banach. Suppose that there is $h \in L^1(0,1)$

s.t. $\forall_{f \in L^\infty(0,1)}$ $\int f h =: \varphi(f)$.

Applying mollifiers, we see that $h = 0$ a.e. Contradiction

B7 $\quad \Phi : \ell^1 \to (\ell^\infty)^*$

$$\Phi : \ell^1 \ni x \longmapsto \left[\ell^\infty \ni y \longmapsto \sum x_i y_i\right]$$

Φ is well-def: fix $x \in \ell^1$. Then

$$|\Phi(x)(y)| \leq \|x\|_1 \|y\|_\infty \text{ by Hölder} \Rightarrow \|\Phi(x)\| \leq \|x\|_1.$$

But there is $\varphi \in (\ell^\infty)^*$ not of this form.

Consider space of convergent sequences c and define

$$\varphi(y) = \lim_{k \to \infty} y_k$$

$\varphi \in (c, \|\cdot\|_\infty)^*$ as $|\varphi(y)| \leq \|y\|_\infty$. Hence, we can extend φ to the whole of ℓ^∞. Suppose that there is $z \in \ell^1$ s.t.

$$\varphi(y) = \sum z_i y_i.$$

We know that $\varphi(e_i) = 0$ (as $e_i \in c_0 \subset c$) so $z_i = 0 \ \forall i$. Contradiction as $\varphi \neq 0$.

Geometric versions of Hahn-Banach

$(E, \|\cdot\|)$ normed space

most of the time $B = \{x_0\}$, A closed subspace

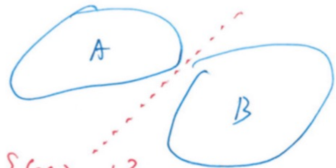
- $A, B \subset E$, one of them is open, convex, nonempty, disjoint.

$\{\varphi(x) = \lambda\}$

$$\sup_{a \in A} \varphi(a) \leq \lambda \leq \inf_{b \in B} \varphi(b)$$

$\boxed{\text{SEPARATION}}$

- $A, B \subset E$, A closed, B compact, both convex, nonempty, disjoint



$\{\varphi(x) = \lambda\}$

$$\sup_{a \in A} \varphi(a) < \lambda < \inf_{b \in A} \varphi(b)$$

$\boxed{\text{STRICT SEPARATION}}$

More practical version of H-B geometric theorems is given below:

C1 Let $x_0 \in E \setminus \overline{F}$ and use H-B strict separation with closed set $\overline{F}$ and compact $\{x_0\}$. Then $\exists_{\varphi \in E^*}, \exists_{\lambda \in \mathbb{R}}$ s.t.

$$\sup_{f \in F} \varphi(f) < \lambda < \varphi(x_0)$$

In particular, $\forall_{f \in F}$ $\varphi(f) < \lambda < \varphi(x_0)$. As F is subspace $\varphi(f) < \lambda \Rightarrow \varphi|_F = 0$ (we can scale elements in F).

But then $0 < \lambda < \varphi(x_0)$. Finally, we take $\dfrac{\varphi(x)}{\|\varphi\|}$.

$\Downarrow$

$\varphi \neq 0$

$\square$.

(C2) Suppose that F is not dense. Then $\overline{F} \neq E$ and applying C1 we get $\varphi|_{\overline{F}} = 0$ but $\varphi \neq 0$. Contradiction

<u>Remark</u>: this is used to prove that a subspace is dense.

(C3) H – Hilbert space, $M \subset H$ linear subspace.

$$(M^\perp)^\perp = \overline{M}$$

$\supseteq$: Let $x \in M$. We want $x \in (M^\perp)^\perp$ i.e. $\forall_{y \in M^\perp}$ $\langle y, x \rangle = 0$. But this is true as $x \in M$.
As $(M^\perp)^\perp$ is closed $(M^\perp)^\perp \supset \overline{M}$.

$\subseteq$: suppose there is $x \in (M^\perp)^\perp$ s.t. $x \notin \overline{M}$.
We can separate x and $\overline{M}$ to get for some $\varphi \in H^*$

$$\sup_{y \in \overline{M}} \varphi(y) < \lambda < \varphi(x)$$

there is g s.t. $\varphi(y) = \langle g, y \rangle$

$$\Rightarrow \sup_{y \in \overline{M}} \langle g, y \rangle < A < \langle g, x \rangle \Rightarrow$$

As $\sup_{y \in \overline{M}} \langle g, y \rangle \leq \lambda \Rightarrow \langle g, y \rangle = 0 \ \forall_{y \in \overline{M}} \Rightarrow$

$g \in M^{\perp}$. But $x \in (M^{\perp})^{\perp}$ so $\langle g, x \rangle = 0$.

We get $0 < \lambda < 0 \Rightarrow$ CONTRADICTION.

C5 Riesz Lemma:

Let M be a closed subspace. For each $\alpha \in (0,1)$, there is $y \in X \setminus M$ such that $\mathrm{dist}(y, M) \geq \alpha$. ($\|y\| = 1$).

Proof: Clearly, there is φ s.t. $\varphi|_M = 0$, $\|\varphi\| = 1$. Choose y s.t. $\varphi(y) \geq \alpha$. Then

$$\alpha \leq \varphi(y) = \varphi(y - x) \leq \|\varphi\| \, \|y - x\| = \|y - x\| \quad \square.$$

C7 Ball is <u>not</u> compact.

Choose any vector x_1 s.t. $\|x_1\| = 1$. Let $Y_1 = \mathrm{span}\{x_1\}$. Use Riesz Lemma to find x_2 s.t. $\|x_2\| = 1$. and $\mathrm{dist}(x_2, Y_1) \geq \frac{1}{2}$. By induction, we find sequence $\{x_i\}$ s.t. $\|x_i\| = 1$ and $\|x_i - x_j\| \geq \frac{1}{2}$, $i \neq j$.

Clearly, $\{x_i\}$ cannot have convergent subsequence.

(8) (A) Indeed, X and Y are closed as convergence in ℓ^1 implies convergence of sequence elements.

To check that $\overline{X+Y} = \ell^1$, we prove that $e_i = (0, 0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots)$ belongs to $X+Y$ (this is sufficient as we know that sequence $\{e_i\}_i$ forms Schauder basis of ℓ^1 so $\overline{\operatorname{span}(e_1, e_2, \dots)} = \ell^1$.

Note that $e_1, e_3, e_5, \dots \in X$ so $e_1, e_3, e_5 \in X+Y$ (we take element from Y to be 0. Note that the sequence $y^n = \{y_i^n\}$

$$y_i^n = \begin{cases} 1 & \text{on } (2n-1) \text{ position} \\ \frac{1}{2^n} & \text{on } 2n \text{ position} \\ 0 & \text{otherwise} \end{cases} \in Y.$$

Then $$e_{2n} = \underbrace{2^n \cdot y^n}_{\in Y} - \underbrace{2^n \cdot \underbrace{e_{2n-1}}_{\in X}}_{\in X} \in X+Y.$$

✓

(B) Sequence c is defined with $c_{2n-1} = 0$, $c_{2n} = \frac{1}{2^n}$.

Suppose $c = x+y$ where $x \in X$, $y \in Y$.

$$c_{2n} = x_{2n} + y_{2n} \Rightarrow c_{2n} = y_{2n} = \frac{1}{2^n} \Rightarrow y_{2n-1} = 1 \Rightarrow y \notin \ell^1.$$

contradiction

C) $Z = X - c$. $Z \cap Y = \emptyset$: Let $u \in X - c$, $u \in Y$. $\Rightarrow$

$\exists_x \; u = x - c \Rightarrow c = \underbrace{x}_{\in X} - \underbrace{u}_{\in Y} \Rightarrow c \in X+Y$ contradiction

Suppose there is $\varphi \in (\ell^1)^*$ s.t. for some $\lambda \in \mathbb{R}$

$$\varphi(z) < \lambda < \varphi(y) \quad \forall_{z \in Z} \ \forall_{y \in Y}$$

As $Z = X - c$ we can write

$$\varphi(x) - \varphi(c) < \lambda < \varphi(y) \quad \forall_{x \in X} \ \forall_{y \in Y}$$

But this shows

$$\begin{aligned} \varphi(x) &< \lambda + \varphi(c) \quad \forall_{x \in X} \\ \varphi(y) &> \lambda \quad \forall_{y \in Y} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \varphi|_X &= 0 \\ \varphi|_Y &= 0 \end{aligned}$$

$\Rightarrow \varphi|_{X+Y} = 0$. As $\overline{X+Y} = \ell^1$ we have $\varphi|_{\ell^1} = 0$

$\Rightarrow$ contradiction as φ is 0 functional.