Functional Analysis, PS 8

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1 $\{e_\alpha\}_{\alpha\in A}$ OB $\iff \left(\forall_x \ \langle e_\alpha, x\rangle = 0 \Rightarrow x = 0\right)$.

($\Rightarrow$) If $\{e_\alpha\}_{\alpha\in A}$ OB, $\forall_x$ $x = \sum_{\alpha\in A}\langle e_\alpha, x\rangle e_\alpha$.

Suppose that $\langle e_\alpha, x\rangle = 0$ $\forall_\alpha$. Then, indeed $x = 0$.

($\Leftarrow$) Suppose $\{e_\alpha\}_{\alpha\in A}$ is not OB, i.e. there is orthonormal set $\{e_\alpha\}_{\alpha\in\bar{A}} \supset \{e_\alpha\}_{\alpha\in A}$. Then,

In particular, there is some f, $\|f\| = 1$ and $f \notin \{e_\alpha\}_{\alpha\in A}$ but f is $\perp$ $\{e_\alpha\}_{\alpha\in A}$. It follows that $\langle f, e_\alpha\rangle = 0$ but $f \neq 0$. Contradiction.

In this way it was proved that $\{e^{ikx}\}$ is an orthonormal basis of $L^2(0, 2\pi)$.

2. $\{e_\alpha\}_{\alpha\in A}$ OB $\Longleftrightarrow$ $H = \overline{\text{span}\{e_\alpha : \alpha\in A\}}$.

($\Rightarrow$) $x = \sum_{\alpha\in A} \langle x, e_\alpha\rangle e_\alpha$ and this series is conv. in H so each x in H can be written as an element of $\overline{\text{span}\{e_\alpha : \alpha\in A\}}$.

($\Leftarrow$) For if not, there is $f \perp \{e_\alpha\}_{\alpha\in A}$, $\|f\|=1$.
As $f \perp \{e_\alpha\} \Rightarrow f \perp \text{span}\{e_\alpha\} \Rightarrow$

$f \perp \overline{\text{span}\{e_\alpha\}} \Rightarrow f \perp H \Rightarrow f = 0$.

Pf of ①: We want $\langle f, x\rangle = 0 \quad \forall_{x\in \overline{\text{span}\{e_\alpha\}}}$.
There is $x_n \in \text{span}\{e_\alpha\}$, $x_n \to x$. We have $\langle f, x_n\rangle = 0$. So $\langle f, x\rangle = \langle f, \lim_{n\to\infty} x_n\rangle = \lim_{n\to\infty}\langle f, x_n\rangle = 0$.

③ Using Parseval's identity $(t \in (0,1))$

$$\sum_{n=1}^{\infty} \left| \int_0^t x^3 f_n(x)\,dx \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \mathbb{1}_{x\in(0,t)}\, x^3 f_n(x)\,dx \right|^2$$

$$= \sum_{n=1}^{\infty} \left\langle \mathbb{1}_{x\in(0,t)}\, x^3, f_n(x) \right\rangle^2 \underset{\text{OB}}{=} \left\| \mathbb{1}_{x\in(0,t)}\, x^3 \right\|_2^2$$

$$= \left| \int_0^t x^6\,dx \right|^2 = \left(\frac{1}{7} t^7 \right)^2.$$

⑤ H-separable. Let $\{x_k\}_{k\geq 1}$ countable dense set. Using Gram-Schmidt we construct $\{y_k\}$ which is orthonormal set in H and

$$\overline{\operatorname{span}(x_1, x_2, \dots)} = \overline{\operatorname{span}(y_1, y_2, \dots)}$$

We have $H = \overline{(x_1, x_2, \dots)} = \overline{\operatorname{span}(x_1, x_2, \dots)} =$
$= \overline{\operatorname{span}(y_1, y_2, \dots)}$.

$\Rightarrow \{y_i\}_{i\in\mathbb{N}}$ basis of H.

Per Enflo

(6) $H = \ell^2$. (for H - separable)

$T : H \to \ell^2 \quad Tx = (\langle x, e_1\rangle, \langle x, e_2\rangle, \dots)$.

- injective: $Tx = 0 \Rightarrow \langle x, e_i\rangle = 0 \ \forall i$
  $\Rightarrow x = 0$ (by (2)).
- surjective: $y \in \ell^2$, $x = \sum y_i e_i$. Then, we have $Tx = y$.
- $\|Tx\|_{\ell^2} = \|x\|_H$

$$\|Tx\|_2^2 = \sum \langle x, e_i\rangle^2 = \|x\|_H^2 .$$

✓.

(7) $v_0 = 1$, $v_n = \operatorname{sgn}(\sin(2^n \pi t))$

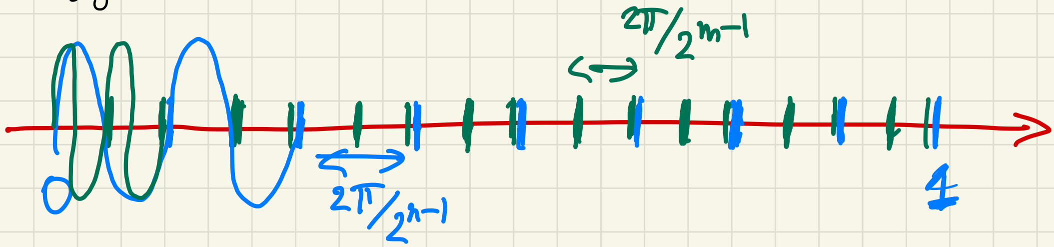
This is orthonormal set. Indeed $\int v_n^2 = 1$.

Moreover,

$$\int_0^1 1 \cdot \operatorname{sgn}(\sin(2^n \pi t)) = 0$$ as sin spends the same time above and below zero.

Finally, $\langle r_n, r_m \rangle =$

$$= \int_0^1 \operatorname{sgn}(\sin(2^n \pi t))\, \operatorname{sgn}(\sin(2^m \pi t))\, dt$$



$\sin(2^n \pi t)$ oscillates $\dfrac{2^n \pi}{2\pi} = 2^{n-1}$

$\sin(2^m \pi t)$ oscillates $\dfrac{2^m \pi}{2\pi} = 2^{m-1}$

$m > n.$

Let $A_i = \left[(i-1)\dfrac{2\pi}{2^{n-1}}, i\dfrac{2\pi}{2^{n-1}}\right]$, $\bigcup_{i=1}^{2^{n-1}} A_i = [0, 2\pi]$

$$\int \dots = \sum \int_{A_i} \dots = \sum \left[\int_{A_i^+} \dots + \int_{A_i^-} \dots \right]$$

On both A_i^+, A_i^- $\sin(2^n \pi t)$ has constant sign.

But $\sin(2^m \pi t)$ on each of this subintervals spends the same time above and below.

$\Rightarrow \{r_i\}$ is orthonormal system

But it is not basis. Consider $r_1 r_2$. We claim that $(r_1 r_2, r_i) = 0 \ \forall i$ but $r_1 r_2 \neq 0$.

For $i=1$, $i=2$ it is clear. For $i \geq 3$

$\int r_1 r_2 r_i = 0$ by the same method as above.

(8) !!! $\langle x, e_i \rangle \to 0 \quad i \to \infty$.

This follows from Bessel's inequality.

Π: Various applications of this fact.

(13) Let H be separable HS, $\{x_n\}_{n\geq 1} \subset H$ bdd. Then $\{x_n\}_{n\geq 1}$ has a subsequence converging weakly to some $x \in H$.

PROOF: We need to find subsequence such that $\langle x_{n_k}, y \rangle \to \langle x, y \rangle$ $\forall_{y \in H}$ for some $x \in H$.

First, we prove it for $y = e_i$, $i = 1, 2, \dots$. We can always find subsequence such that $\langle x_n^{(1)}, e_1 \rangle$ is convergent to some $a_1 \in \mathbb{R}$. Further we can find subsequence of the found subsequence $x_n^{(2)}$ s.t.

$$\langle x_n^{(2)}, e_2 \rangle \to a_2 \qquad n \to \infty$$
$$\langle x_n^{(2)}, e_1 \rangle \to a_1 \qquad n \to \infty.$$

We construct family of subsequences

$$x_n \supset (x_n^{(1)}) \supset (x_n^{(2)}) \supset \dots$$

such that $\langle x_n^{(i)}, e_k \rangle \to a_k \quad \forall_{k = 1, \dots, i}$.

To obtain one seqcence, we define

$$y_n = x_n^{(n)}.$$

Then, $(y_n, e_i) \to a_i \quad \forall_{i \in \mathbb{N}}$ (because starting from $n=i$, $y_n \subset x^{(i)}$).

This is called "diagonal procedure".

We want $a_i = \langle x, e_i \rangle$ for some $x \in H$.
We can take $x = \sum_{i=1}^{\infty} a_i e_i$. But this won't work (we don't know anything about convergence of this series).

We proceed differently. First, consider $G = \operatorname{span}(e_1, e_2, \dots)$. On G we can define

$$\varphi(x) = \lim_{n \to \infty} (y_n, x)$$

φ is bounded on G: $|\varphi(x)| \leq \sup \|y_n\| \, \|x\| \leq \left(\sup_{n \in \mathbb{N}} \|y_n\| \right) \|x\|$.

As $H = \overline{G} \Rightarrow \varphi$ has a unique extension to H denoted as $\tilde{\varphi}$ and by RRT $\exists_z$ such that
$\varphi(x) = (z, x)$.

Coming back to G we see that

$$\langle y_n, e_i \rangle \longrightarrow \langle z, e_i \rangle \qquad n \to \infty$$

As $\overline{\operatorname{span}(e_1, e_2, \dots)} = H$ we have

$$\langle y_n, x \rangle \longrightarrow \langle z, x \rangle \quad \forall_{x \in H}. \quad (\pi)$$

So that $y_n \rightharpoonup z$.